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BIOMETRIKA

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FOUNDED BY

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"Biometrika" Portrait Series, No. IV. Issued with Vol. XVIII, Parts III and IV.

VOLUME XVIII NOVEMBER, 1926 PARTS III AND IV

ON THE SKULL AND PORTRAITS OF GEORGE BUCHANAN.

BY KARL PEARSON, F.R.S.*

I EXPECT that many of those present will consider it an act of great rashness that a southerner should venture into the metropolis of northern Britain and dare, a mere scientist, to talk about a man whom many Scots look upon as Scotland's greatest scholar, leading historian and the first writer in Scotland to propound a political philosophy.

But I would remind you that for good or ill we cannot separate the history of our two countries. Buchanan played a large part—a larger part than some of us recognise—in those still obscure proceedings which, starting at Langside, were continued at York and Hampton Court, and culminated only in the final scene at Fotheringay. Mary Stewart's history is as much English as Scottish history, and the story of both countries would not only have been different, but differently written, had the Tudor blood been as dominant in Mary as in Elizabeth; while the psychological bearing of the extent to which Mary did inherit Tudor characteristics has been often overlooked by the historians of both countries.

The time has come when we are justified in taking, not a heptarchical view of the history of Britain, but a British view, which would recognise that local histories can be interwoven into a wider history of our united countries. Queen Mary, George Buchanan, Darnley, Moray, Bothwell, and the rest of the Scottish actors of those days really played their parts on a wider stage than local history provided; that is not only the source of their perennial interest, but the excuse for my appearance and choice of a subject here this afternoon.

Previous lecturers for the Henderson Trust have told us that while much of the old phrenology is no longer acceptable in the light of modern science, yet increasing acquaintance with the function of brain centres may lead us, not so much from observations on the external surface of the skull as from the markings on the endocranial cast, to suggestions—not to use too emphatic a word—as to character and abilities. I should like at once to state that this is not to be the line of my inquiry this afternoon. I think an endocranial cast of George Buchanan's skull would be of interest, but I have not ventured to take it nor to ask for it to be taken.

I want to put before you something much less abstruse, and more direct in its appeal to your imaginations.

In the rough and tumble of everyday experience, in trusting to our fellow men, in the appointment of colleagues, assistants and servants, and even in the

* Being a lecture delivered on June 4 in the Anatomy Theatre of the University of Edinburgh for the William Ramsay Henderson Trust.

choice of acquaintances and friends, we are largely influenced by our reading of character from physiognomy. We may admit it or we may not, but it is a fact that facial expression draws our confidence or serves as a warning mark. The reading of physiognomy is not a science with its ascertained rules; it is not wholly, I believe, a result of past experience, but rather an instinct modified by experience. We are reduced to the sentiment of Martial, as englished by Tom Brown:

> "I do not love thee, Doctor Fell,
> The reason why I cannot tell;
> But this alone I know full well,
> I do not love thee, Doctor Fell."

Even a Christchurch undergraduate is susceptible to facial expression!

Probably in judging half-unconsciously by physiognomy we are right in nine cases out of ten, and badly wrong on the tenth occasion. We should do better, if we judged by the dynamic expression as given by the film of an individual rather than by the static expression conveyed by a bust or portrait.

The reconstruction from the skull of the head is a branch of science, which has of recent years made some progress, but while the facial expression does to some extent depend on the form of the skull, and the strength of the musculature which can be partially judged from the skull markings, we are left wholly in the dark as to those facial lines which leave no permanent traces on the skull itself, and which have so much to do with our judgment of character.

Why is it that we desire to see and to acquire portraits of great men, or of the individual men we treat as our personal heroes or saints? I think it undoubtedly arises from the fact that a portrait, at any rate a good portrait, helps us to a truer characterisation, a truer appreciation, than many pages of verbal description. We are thrown back on our instinctive judgment of physiognomy. I believe that at present one of the chief services which can be done by aid of the skull of a great man is to find out from it which portrait, if portraits exist, truly represents him. With all due reverence I would exhume the skull of Shakespeare, of Newton or of Milton, and then we should be able to standardise their portraiture, and say: Here is the true man, and what would also be of interest: There is the artistic liar!

Something of this kind has been done in my Laboratory for two historic skulls—that of Sir Thomas Browne and that of Henry Stewart, Lord Darnley; and I may be permitted to indicate here how the method serves to differentiate good from bad portraiture.

On Plate I we have a portrait of Jeremy Bentham fitted with the significant lines of his "Auto-Icon." The mummified head, when the drooping mandible is brought into touch with the upper jaw, gives a quite passable fit, showing that the artist was reasonably truthful. On Plate II we have a contrast between two portraits of Sir Thomas Browne fitted with the outline of his skull adjusted as closely as possible to the two representations. The earlier or Allix portrait agrees well with the skull, while the later portrait, almost certainly painted after his

death for the Royal College of Physicians, exaggerates his low frontal into "high-brow" intellectuality. The artist was quite clearly an incorrigible idealiser.

Can we apply the same method to obtain further light on George Buchanan as judged from his portraits? To attempt this will be the purpose of my lecture this afternoon. But before we proceed further with the portraits, it is wise to inquire what evidence we have for the authenticity of the skull.

Buchanan died on September 28, 1582, aged 76 years.

There is nothing in the skull to contradict this age. It is the calvarium of an old man; there is no mandible, which would have told us something additional, but the teeth in the upper jaw must have been present at death, although the alveolar border has extensively receded.

The bony walls of the skull have been reduced in parts to the thinness almost of a railway ticket in a manner that is very characteristic of extreme age, and familiar to those who have handled large numbers of crania. The zygomatic arches, at least to judge from the one that is complete, could never have been massive and have dwindled with age almost to the thickness of a quill. The owner of the skull could hardly have had prominent cheek bones, except by extreme thinness of the flesh.

Coming to the history of the skull itself, we naturally find its first purloining—as in the case of Sir Thomas Browne's—involved in obscurity. It is the more so, perhaps, in this case, as the purloiner was a man of position in Edinburgh—no other than the Rev. John Adamson, Principal Regent of the University from 1623 to his death in 1653. Adamson was a great admirer of Buchanan, but he was hardly likely to state, during his official life, how and when he had despoiled Buchanan's grave. Puissant as we all recognise the principal of a university to be, humbly as we obey his dictates, yet we perceive that his autocracy—wide as it is—does not include body-snatching as a prerogative. There are some things which even the principal of a university cannot do publicly, but we may be fairly sure that when he does them *sub rosa*, he will carry them through successfully, and obtain exactly what he is seeking. I think we need not doubt that Adamson, having set his heart on Buchanan's skull, was not likely to fail in getting the actual article.

Adamson was educated at St Andrews, being born in the year 1576; he held I think a professorship of philosophy there, but in 1598 became a regent in Edinburgh, holding his chair there till 1604, when he accepted a ministerial call to North Berwick, being afterwards translated to Libberton and returning to Edinburgh as Principal in 1623.

In 1598, George Buchanan had been buried for 16 years, in 1623 for 41 years. We must further note that Buchanan had died in poverty, directing the small sum of money he had left to be given to the poor, and telling his friends to let him lie where he was till the City of Edinburgh buried him. He appears to have been interred without a tombstone. Adamson in his epitaph or eulogy on Buchanan draws attention to this fact, saying that only those need monuments who have

nothing else to be honoured by. A grave without a tombstone is very likely to be encroached upon for later interments, and Adamson was undoubtedly familiar with the situation of Buchanan's grave. Probably he obtained the skull from the grave-diggers at some later interment at or near Buchanan's tomb. Sir Robert Sibbald, born twelve years before Adamson's death, writing of Adamson within thirty years of the latter's death, tells us, somewhat naively, that by his, Adamson's, care the skull of Buchanan, dragged from its grave, has been preserved in the Library of the University of Edinburgh. There it remained, accompanied by a copy of Adamson's Latin verses, to which I have already referred, till 1817, when it was transferred to the Museum of Anatomy. About 100 years after Buchanan's death Mr Robert Henderson, University Librarian, refers to the skull, and says of Adamson: "It 's storied of him that being at Buchanan's burial, he bargained for the skull; which having got from the graveman within the year or so, he carefully kept in his life-time; and being found in his study so inscribed after his removal was handed down to us." A child of six might be, if improbably, at the burial of Buchanan, but he certainly would not have bargained with the graveman for Buchanan's skull on that occasion. However the main point of Henderson's account, that the skull was found in Adamson's study after his death, inscribed with the name of Buchanan, doubtless corresponds to the truth. And if so, Adamson would hardly have obtained the skull without the assistance of the sexton, or kept it without a fair amount of certainty that it came from George Buchanan's tomb.

It is noteworthy that while John Adamson says there was no tombstone to Buchanan's grave, we read in 1701 of the existence of such a stone.

"At Edinburgh, the 3rd day of December 1701, the same day the council being informed, that the through stone [tombstone] of the deceast George Buchanan lyes sunk under the ground of the Greyfriars: therefore they appoint the chamberlain to raise the same, and clear the inscription thereon; so that the same may be legible."

If we could ascertain at what date that stone was erected, we should have an upper limit to the date when the Principal of the University robbed the Greyfriars, for there was clearly no stone when he wrote his verses.

I have not succeeded in finding any real work done on the skull in relation to the portraits.

Mr William Carruthers in the *Scottish Review* for October 25, 1906, dismissed the skull, stating that it could not be that of George Buchanan, because it was not like the portrait in the National Portrait Gallery in London. The question seems to be: Is it like any other portrait? if it is, it would only show that the National Portrait Gallery picture is not a truthful representation of Buchanan.

Sir William Hamilton is said to have tested Buchanan's portraits against the skull about 1835 by measurement and found that *only* the Chouet woodcut (see Plate III) fulfilled his tests. In other words he appears to have rejected the two portraits in the Senate Hall of his own University, which are of the National Gallery type. Speaking of the Chouet wood engraving and its reproduction by

Freiherus, Hamilton writes: "The head in both is thoroughly Scottish in character with a long and well-formed nose, well-defined cheek bones and a long upper lip as in the skull." Now these three characters are all present in the National Portrait Gallery type (see Plate IV), but appear to be quite absent in the skull.

I am not clear what a head "Scottish in character" may be. It is not possible craniometrically to distinguish really significant differences between the Lowland Scottish measured by Turner and the 17th century Londoners measured in my own Laboratory. Recently some 400 crania from a graveyard in London dating round the plague year 1665 have been measured by us and we have obtained the type contours on the average of some 140 male skulls. The upper lip and the nose of this typical skull from the pauper graveyard are identical in length with those of the Buchanan skull, which like these 17th century Londoners, had no specially marked zygomatic arches*. The main difference between the Buchanan skull and that of the 17th century Londoner lies in the very feeble development in the former of the occipital and cerebellar regions.

I do not think we can place any weight on Sir William Hamilton's rejection of all but the Boissard-Chouet engraving, especially as that engraving is in my opinion undoubtedly a copy of the Edinburgh or Bronckhorst type of portrait, a point which Sir William Hamilton seems to have overlooked.

Lastly I may refer to Professor Hume Brown's *Life of George Buchanan*, wherein he writes: "It may be added that what on good authority is supposed to be Buchanan's skull is preserved in the Anatomical Museum of the University of Edinburgh. Its general outline is exactly like that of the best portraits, and by its dome-like shape and extreme tenuity it has all the marks of a high cerebral development."

The skull has certainly extreme tenuity, but why this should be a mark of "high cerebral development" instead of as usual the concomitant of old age, I do not understand. As to its dome-like character, its auricular height is just the average of our Londoners †, and the "dome," such as it is, is produced in appearance by the marked absence of occipital development. Professor Hume Brown does not tell us the locus of the best portraits which the general outline of the skull is exactly like. The fact that Sir William Hamilton and Professor Hume Brown appear to reach opposite conclusions may, perhaps, justify a biometrician in endeavouring to advance further by more elaborate measurements.

I have placed in the accompanying Table in parallel columns the usual series of cranial measurements taken in the Biometric Laboratory, for (i) the average male Londoner of the 17th century, (ii) the late 16th century George Buchanan, (iii) the 17th century author of the *Religio Medici*, and (iv) the early 19th century

* Zygomatic Breadth: Buchanan's Skull, 130 mm.; Farringdon Street Graveyard, 130·1 mm. Facial Breadth: Buchanan's Skull, 92·5 mm. (?); Farringdon Street Graveyard, 91·4 mm. (Moorfields Plague Pit, 93·9 mm.).

† Males: 17th century Londoners. Farringdon Street Poor Graveyard 110·0, Whitechapel Plague Pit 112·1, Moorfields Plague Pit 111·3. These are *mean* values, hence the Buchanan skull, 110·0, is simply mediocre.

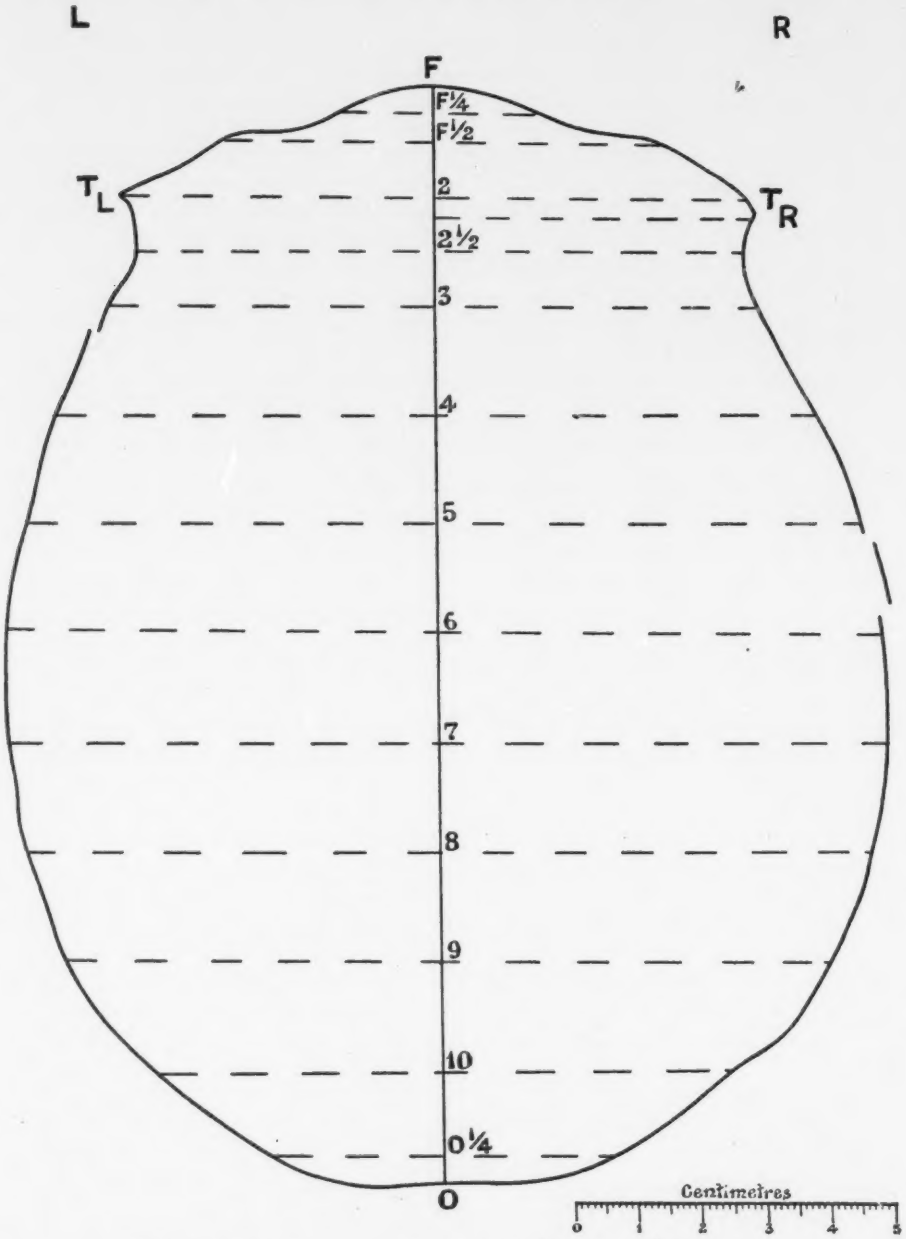
Comparison of the Cranial Measurements of George Buchanan with those of other Skulls.

Cranial Character	Average London Poor 17th Century ♂s	Skull of George Buchanan	Skull of Sir Thomas Browne	Skull of Jeremy Bentham	Deviation of Buchanan skull from 17th Century ♂s. Difference / S.D.
Capacity	1481	1360 (??)	1509	1475	− ·93
Length (Flowers)	186·1	171·3	190·0	183·5	**−2·75**
Length (Maximum)	188·8	174·0	194·6	186·0	**−2·29**
Breadth (Parietal)	142·4	140·0	143·7	147·0	− ·41
Breadth (Minimum frontal)	96·8	98·8	87·4	99·0	+ ·34
Height (Basio-vertical)	130·4	130·6 (?)	122·0	—	+ ·04
Height (Basio-bregmatic)	129·7	129·9 (?)	121·9	132·0	+ ·04
Height (Auricular)	110·0	111·0	102·6	116·0	·00
Skull Base	100·1	100·9 (?)	102·7	102·0	+ ·18
Arc, Transverse (Apex)	309·0	304·0	297·0	325·0 (?)	− ·39
Arc, Sagittal	378·8	355·4	380·0	365·0 (?)	**−1·65**
Arc (Nasion to Bregma)	129·3	119·5	129·0	130·0 (?)	**−1·51**
Arc (Bregma to Lambda)	128·1	124·0	123·0	120·0 (?)	− ·51
Arc (Lambda to Opisthion)	120·6	111·9 (? Reconst.)	128·0	115·0 (??)	**−1·11**
Chord (Lambda to Opisthion)	97·3	91·3 (? Reconst.)	99·8	97·1 (?)	**−1·17**
Horizontal Circumference	530·0	503·0	549·0	540·0 (?)	**−1·70**
Alveolar Point to Nasal Spine	19·2	21·5	21·7 (?)	22·0 (??)	+ ·82
Facial Height	70·5	69·6	73·1 (?)	73·0 (?)	− ·20
Facial Breadth	91·4	92·5 (?)	90·9	97·0 (??)	+ ·18
Bizygomatic Breadth	131·0	130·0	132·3	127·0 (?)	− ·21
Nasal Height, R.	51·8	51·0	52·2	52·8	− ·28
Nasal Height, L.	51·7	51·7	52·2	52·8	·00
Nasal Breadth	24·6	25·6 (?)	22·6	26·7	+ ·40
Dacryal Subtense	12·8	12·0	12·8	11·9 (?)	− ·45
Dacryal Chord	22·2	22·2	20·9	22·2 (?)	·00
Dacryal Arc	36·1	32·0	32·7	33·0 (?)	**−1·15**
Simotic Subtense	4·6	4·5	5·0	4·3	− ·09
Simotic Chord	9·2	10·4	8·7	9·5	+ ·63
Breadth, R. Orbit	42·3	45·8	43·7	45·2 (?)	**+2·24**
Breadth, L. Orbit	42·4	44·8	42·8	46·0 (?)	**+1·62**
Height, R. Orbit	34·3	35·9	36·3	36·0 (?)	+ ·68
Height, L. Orbit	34·3	36·8	36·7	36·0 (?)	**+1·09**
Palate Height	11·1	10·5	16·5	12·0 (?)	− ·42
Palate Breadth	39·3	36·8	37·5 (??)	35·0 (?)	− ·82
Palate Length to base of Spine	46·0	38·7 (??)	49·0 (??)	47·0 (?)	**−2·75**
Profile Length	94·4	89·5 (?)	98·3 (?)	89·0 (?)	− ·92
Length, Foramen Magnum	36·8	32·6 (?? Reconst.)	34·6	36·2 (?)	**−1·40**
Breadth, Foramen Magnum	30·6	—	30·1	31·0 (?)	—
Profile Angle	85°·9	87°·8	84°·7 (?)	93°·0 (?)	+ ·60
Angle at Nasion	64°·2	60°·5 (?)	65°·4	58°·0	**−1·01**
Angle at Basion	42°·5	42°·3 (?)	42°·7 (?)	44°·5	− ·05
Angle at Alveolar Point	73°·3	77°·2 (?)	71°·9 (?)	77°·5	**+1·07**
1st Cephalic Index (100 B/L)	75·4	80·5	73·8	79·0	**+1·47**
2nd Cephalic Index (100 H/L)	69·3	75·1 (?)	62·7	70·9	**+1·80**
3rd Cephalic Index (100 B/H)	109·1	107·2 (?)	117·8	111·4	− ·40
Compound Cephalic Index (100 $(B-H)/L$)	6·7	5·4 (?)	11·2	8·1	− ·36
Facial Index	77·1	75·2 (?)	80·4	75·3 (?)	− ·30
Nasal Index, R.	47·5	50·2	43·3	50·6	+ ·63
Nasal Index, L.	47·5	49·5	43·3	50·6	+ ·46
Foraminal Index	83·4	—	85·7	85·6	—
Dacryal Index	58·1	54·1	61·2	53·6 (?)	+ ·43
Simotic Index	50·7	43·3	57·5	45·3 (?)	+ ·57
1st Palate Index (Breadth/Length)	85·4	95·1 (??)	76·5 (??)	74·5	**+1·75**
2nd Palate Index (Height/Breadth)	28·5	28·5	33·7 (??)	34·3 (?)	·00
Occipital Index	58·0	58·3 (?)	55·8	61·3	+ ·11
Orbital Index, R.	81·0	78·4	83·1	79·6 (?)	− ·43
Orbital Index, L.	80·9	82·1	85·7	78·3 (?)	+ ·23
External Orbital Width	98·1	98·5	98·0	100·4 (?)	+ ·10

Jeremy Bentham. None of these skulls is in anything like perfect condition, and that of Jeremy Bentham is in the mummified head, so that many measurements are difficult to take upon it, and are marked in the table with queries and double queries. Sir Thomas Browne's skull is in a more complete condition than Buchanan's, but still owing to breakages certain measurements are doubtful. In the case of Buchanan, a portion of the left orbit, a considerable portion of the occipital bone and the whole of the foramen magnum have disappeared. There is no mandible, and accordingly I have not given the mandibular measurements for the other crania. In the case of Buchanan I have given measurements obtained by a reconstruction in plasticene of the base of the skull. I do not propose to enter fully into the methods adopted in reconstructing parts of a skull. They can be carried out with more or less exactness, when we know the correlations of its component bones. As an illustration I may take the reconstruction of the basion or anterior point of the foramen magnum. We have first to measure as far as is possible the capacity of the skull. After closing the orifices this was found by mustard seed packing to be 1360 cms.^{3}, very small as compared with the average English, or either Sir Thomas Browne's or Bentham's skull. Now we have a formula for finding the capacity from maximum length, parietal breadth and auricular height. This formula gave a capacity of 1375 cms.^{3}, quite as close to the capacity obtained by mustard seed as could be hoped for. Assuming Buchanan's capacity as known, we can compute the height from the bregma to the basion because we have a second formula giving the capacity in terms of maximum length, parietal breadth and basio-bregmatic height. Proceeding in this way one is able to determine reasonably accurate positions for points on the missing part of the base of the skull, and fill in the portion broken away with plasticene. That is what is meant by the word "Reconstruction" against certain measurements in the table.

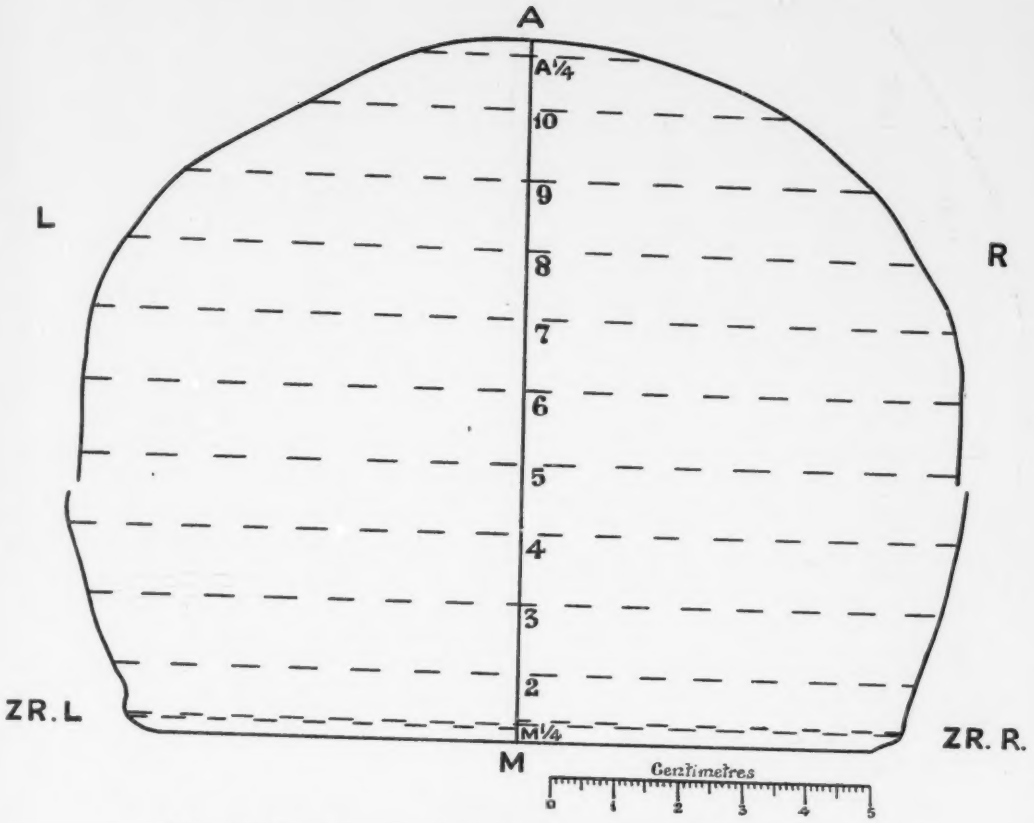
We have seen that Buchanan's skull had a small capacity, and it is at once clear why it had. The height of the skull and its breadth are not small, they are simply mediocre, but the length is remarkably small. Sir Thomas Browne compensated his low frontal and small auricular height by a great length and an increased parietal breadth. But nothing of this kind occurs in Buchanan's skull. It is as if the occipital region and the cerebellum had been simply cut away. This can be shown at once by superposing the horizontal contour of Buchanan's skull on the type or average contour of 140 male English skulls*. You will remark at once nearly complete correspondence in the frontal region, but a total absence of occipital development in Buchanan's skull. You may see the same point in the other two main contours. In the transverse or vertico-auricular section Buchanan is very nearly simple British, he might easily have been a 17th century Londoner; in the sagittal section he is also mediocre British until we come to the occipital, and here comparison is not feasible, because in his skull that region may be said to fail entirely.

* Tissues of the average Farringdon Street contours accompany Miss Beatrix Hooke's paper in the preceding part of this journal and are reproduced with this part.



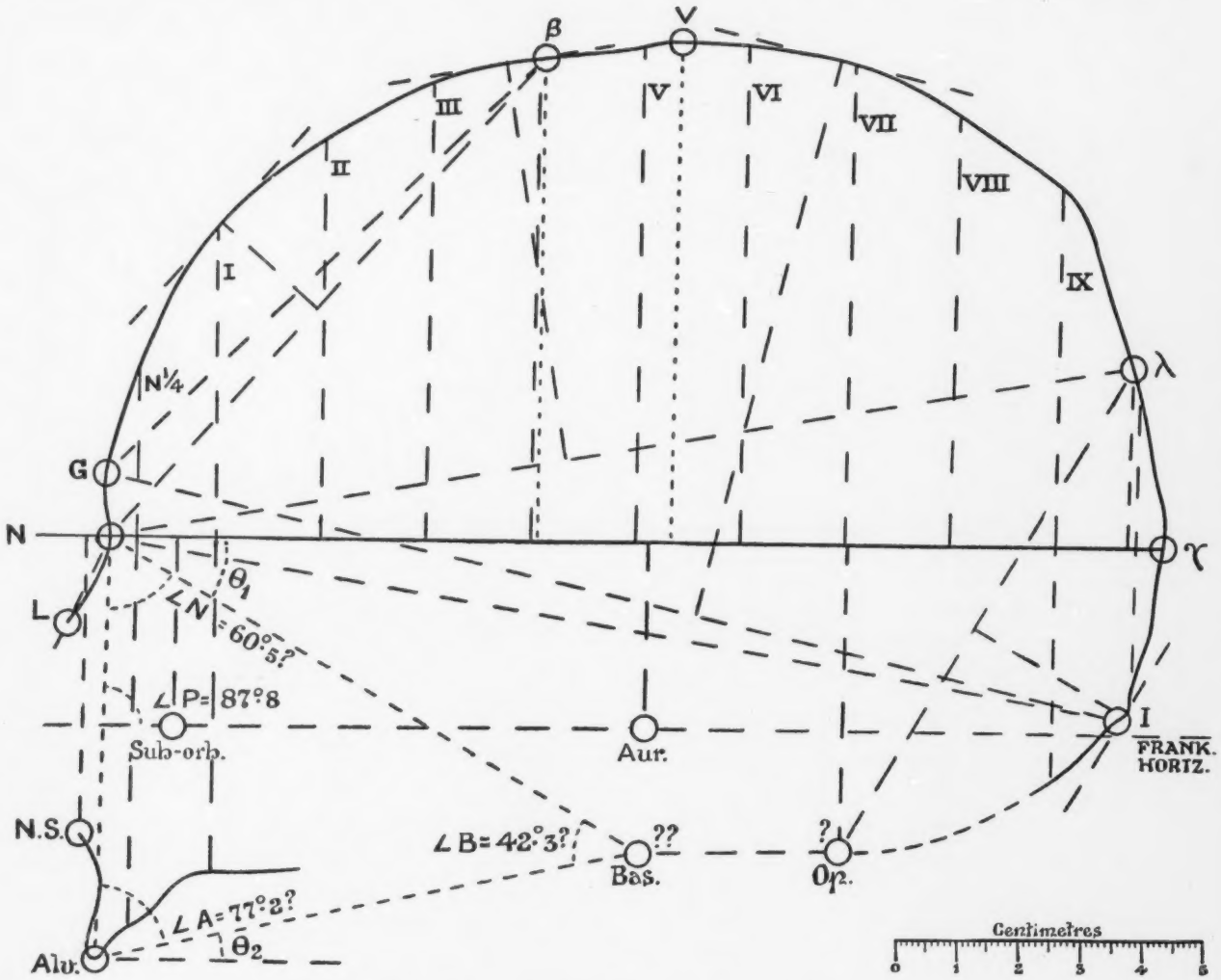
George Buchanan's Skull, Horizontal Contour.

Fig. I.



George Buchanan's Skull, Transverse Contour.

Fig. II.



George Buchanan's Skull, Sagittal Contour.

Fig. III.

Now let us examine the characteristics of Buchanan's skull by another method, namely by taking the difference between the character in Buchanan's case and the corresponding average value in English crania, and dividing this difference by the average variability of the character in the English population. When this ratio is:

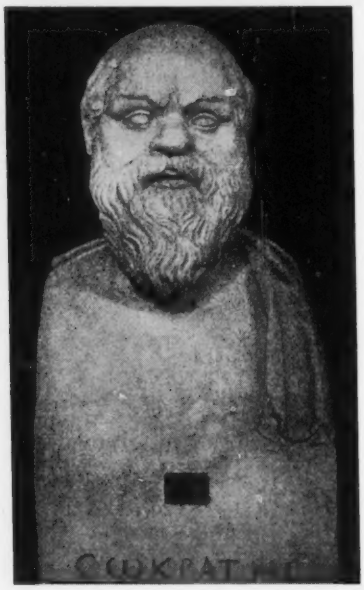
1,	1	person	in	6	has the	characteristic	more emphasised.
1·5,	1	„	„	14 to 15	„	„	„
2,	only 1	„	„	50	„	„	„
2·5,	only 1	„	„	100	„	„	„

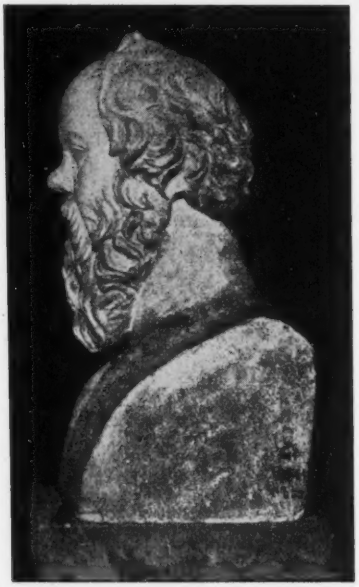
I think a characteristic can hardly be spoken of as individually remarkable until it occurs only once in at least 50 or more individuals. Values of 1 to 2 may be said to contribute somewhat to individuality, but not to be markedly personal features. Now let us look down the fifth column of our Table on p. 238 and see the points in which the owner of this skull possessed individuality. We may examine first certain points in which he certainly had no individuality. I refer to those mentioned by Sir William Hamilton. George Buchanan had certainly not a long nose, it was just the nose of the average man we meet in the street; he had not a long upper lip, one man at least in every five we see has a longer lip than George Buchanan had; again he had not well-defined cheek bones, for his facial breadth and bizygomatic breadth are just normal. In addition he had not a high forehead, his cranial height, whatever estimate we take of it, is simply mediocre, and his frontal bone has precisely the shape of the average British frontal bone. The skull in no particular corresponds with that of a man of high brow, long nose, long upper lip and well-defined cheek bones.

Now let us see the points in which the owner of the skull was individual. First we see from the $-2{\cdot}75$ and $-2{\cdot}29$ that he was individual in the extreme shortness of his head. Next we see that the sagittal arc and its components were short, and that the horizontal circumference was small; this of course follows from the shortness of the length, but the individuality is by no means so marked. In other words the arcs have made up for some of the deficiency of the occipital region. We may express this by saying that the skull, if small, is "well-filled." The idea indicated by "well-filled" is the following: Suppose we made a box of the exact length, height and breadth of the skull, and placing in it a bladder, blew air into the bladder until it *just* touched all six sides of the box. That would *not* be a "well-filled" bladder. Now continue to blow air into it; its three principal diameters would be forced to remain the same, but the sagittal, transverse and horizontal arcs would continue to increase and compensate for the smallness of the principal diameters; such a bladder would then be "well-filled." This is the best interpretation I can place on the term a "well-filled" skull*.

* It must be noted, however, that if the capacity be reconstructed from the three principal arcs (see *Biometrika*, Vol. III. p. 370) its value is found to be 1342·5, actually *less* than that reconstructed from the diameters (L, B, OH), i.e. 1372·7. The mean of these two values (1358) is very close to the value obtained by packing with mustard seed.

The next value in excess of unity is that for the dacryal arc. Buchanan's dacryal chord is just the average, but the dacryal arc is short. He must therefore have had a flat bridge to his nose. You will remember this type of nose occurs in the busts of Socrates. I think we must not on this account rush to the conclusion that Buchanan shared the wisdom of Socrates. Indeed modern medicine attributes such noses to courses of conduct hardly wise. But the point I would have you bear in mind is that a picture which has a narrow nasal bridge cannot be a true portrait of the owner of this skull.





Passing down our last column (p. 238) we see that the next value which attracts our attention is the size of the orbits; these, notably the breadth of the right orbit, are marked characteristics, especially when we consider that the skull is small. We must look out in the portraits for large orbits, especially the right one.

The next character which attracts our attention is the markedly short palate; but I think we must proceed here with great caution. I have marked the measurement with a double query, the palate is too much broken to have a good reconstruction, and the skull being short we might assume it to have a short palate. I am not certain of this however, for the auricular point is much in the average position, and accordingly the position of the basion as reconstructed cannot be far out, so that the skull base must be near the normal length, and the palate would not necessarily have been so short. The fact is that the chief peculiarity of Buchanan's skull lies entirely in the occipital region, and as there is no profile

portrait this peculiarity, and naturally that of the palate, cannot be of service for the portraiture.

We now pass to the length of the foramen magnum. This is again a doubly-queried measurement, but I think the foramen was very small; as it is of no service for portraiture we may anyhow pass it by. In going further down our column we notice some slight individuality in the fundamental triangle, but it is only a "one-man-in-six" individuality and may be entirely due to a small error in the reconstruction of the basion. We again need not stay over this point.

Next we reach the two cephalic indices, which are naturally in excess, since the height and breadth are normal, while the length is very short. A cephalic index of 80·5 would be about 83 on the living individual. Out of 3377 living British persons I found 249 or 7·4 % with a cephalic index of 83·5 or upwards. This is about 1 in 14 to 15, precisely what I have said before was indicated by a figure of 1·5 in the last column of our table. Now a cephalic index of 83 for the living head is by no means uncommon in Scotland. Buchanan is said to have sprung from Killearn in Stirling, which according to Dr Tocher's measurements of Scottish soldiers is the most long-headed district of Scotland (mean cephalic index 77·3). His name, however, if we discard the mythical Irishman, suggests a north-eastern rather than mid-western descent. But the best of Aberdonians cannot claim an average cephalic index of more than 78·4, in practical accordance with that of Britishers south of the Tweed. Whence George Buchanan's round-headedness came we cannot say, so ignorant are we of many of the lines of his ancestry. It is possible, but by no means needful, to suppose foreign blood*. Unfortunately this very marked feature of the skull is of no avail for determining portraiture. As far as portraiture is concerned we are left with facial characters only. Most of these characters possess no marked individuality, they are close to those of the average British skull. But there are certain characters which are individual. We should not expect to find a long nose with a narrow bridge, but an average length of nose with a flattened bridge. We should expect not a long upper lip but one of moderate dimensions. We should not expect emphasised cheek bones, but rather a rotundity of face, and this would be surmounted by a hemispherical skull cap suggesting no great height of the forehead. There is nothing in the skull whatever to suggest a long oval face. George Buchanan was certainly *not* a "high brow."

The skull itself will give you the best idea of these points (see Plates V to IX). It is difficult now-a-days to speak of a bullet-headed man, for the shape of a bullet

* Professor Hume Brown says Buchanan came of the happy mixture of Celtic and Teutonic blood. If Celtic origin for Buchanan be assumed because one of his distant ancestors is said to have come from Ireland, and he himself could speak Gaelic, we must anthropologically assert that there is no such thing as a "Celtic race," only a mass of men of very different head shapes, whose ancestors spoke the same family of languages. As for "Teutonic," it is again a linguistic and not a racial term. Of a brachycephalic German ancestor—to account for Buchanan's round-headedness and absence of occipital—I can find no trace in such pedigrees of the Heriots as exist. If by "Teutonic," we are merely to understand Lowland Scottish, these are identical with our Londoners and, so far from being essentially Anglo-Saxon, have for their nearest congeners the men of the Iron Age in Britain.

has changed, but we may I think safely say the owner of this skull had an apple-shaped rather than a pear-shaped head.

With this preliminary consideration of the skull we now turn to the portraiture. As in the case of many men of distinction there exist several incompatible types of portraiture, so for Buchanan there are divergent pictures which undoubtedly cannot all represent him. Those who start with the idea that a particular portrait is certainly that of Buchanan, citing written evidence for it of a certain grade of probability, but then find this portrait does not fit the skull, ought, I hold, to do one of two things:

(i) to consider whether the evidence for the authenticity of the picture is more or less reliable than the evidence for the authenticity of the skull;

(ii) to consider whether the skull is more appropriate to other portraits which have, perhaps, less evidence of their authenticity, but which may after all represent the Simon Pure.

What they certainly must not do is to disregard the skull straight away because it does not accord with the portrait they have selected as giving their own ideal of George Buchanan. That I take it was the line adopted by Mr William Carruthers, who simply dismissed the skull because it was not like one type of presumed portrait. Nor must anyone on the other hand, overlooking the fact that craniometry is a definite branch of science, assert that the skull possesses attributes which actually do not belong to it. A good illustration of this occurs in Dr Aeneas Mackay's article on George Buchanan in the *Dictionary of National Biography*. He writes:

"Tradition dating from a short period after his [Buchanan's] death ascribed to him the skull preserved in the Anatomy Museum of the University, of which there is a print in Irving's life, and which certainly resembles the best authenticated portraits of him which have been preserved, that by Boinard, engraved in Beza's 'Irones' and of which a copy is in the University of Edinburgh."

It is difficult to imagine more errors than are crammed into the last three lines of this statement, overlooking the grammatical blunder, by which "the best authenticated portraits" is suddenly reduced to the singular as "that by Boinard."

Now let us first be quite sure of our dates. George Buchanan died in 1582, aged 76. Beza's volume of *Icones* [not "Irones"!] was published in 1580; it contains no engraving of Buchanan, and no biography of Buchanan. In 1581 Beza's *Icones* was translated into French by Simon Goulard; there was no portrait of Buchanan in it.

Only in 1673, more than ninety years after Buchanan's death, Pierre Chouet published an anonymous volume at Geneva entitled: *Les Portraits des Hommes Illustres, qui ont le plus contribué au Restablissement des belles lettres et de la vraye Religion.*

There is no text to this work, beyond the list of portraits, and no explanation of its origin. All the blocks of Beza and Goulard are used, and *many more besides.* Among the latter is one of GEORGE BVCANAN (see Plate III). It has been too

hastily assumed that Beza had a block cut which neither he nor Goulard used, and that this was preserved for 90 years, to be published later in the following century by Pierre Chouet, also at Geneva.

That portraits of John Knox and George Buchanan were sent to Beza is well known. They were sent by Peter Young, who in a postscript to a letter of Nov. 13, 1579, addressed to Beza, wrote "Just as I am signing this letter the painter has opportunely come in, and brought in a box the likenesses of Buchanan and Knox*."

Further, we find in the Scottish Treasury Accounts: "To Adrian Vaensome, Fleming, painter, for twa picturis, and send to Theodorus Besa, conforme to ane precept, as the samin producit upon compt beris £8. 10*s*."

These pictures must have been painted when Buchanan was 73 years of age. The Beza portrait was not therefore, as Dr Mackay states, painted by Boinard (*sic*), but by Vansom†; it did not appear in Beza's *Icones*, and there is no reason to suppose a copy of that particular portrait is now in the University of Edinburgh.

But let us go a stage further. In 1598 J. J. Boissard (not Boinard, be it noted!), fifteen years after Buchanan's death, issued his *Icones Quinquaginta Virorum illustrium Doctrina et Eruditione.* This is a work independent of Beza's *Icones*, and in this is a portrait of George Buchanan (see Plate III)—"Georgivs Bvchananvs Æta suæ 76." This engraving is by Granthomme; Boissard was the author of the book, not the engraver. Now this engraving in Boissard's *Icones* opens a new field of inquiry. For while the crude engraving in Chouet of 1673 does appear to be the same man as in Boissard, it is not possible to believe that the Boissard copied the Chouet. In the first place there is no evidence that a man working in *Frankfurt* in 1598 would have access to an unpublished engraved block of Buchanan in Geneva, even supposing the block was at that time in existence. Further, and what is more important, the Boissard portrait is clearly a copy of the Bronckhorst type of portrait, and it bears the same age, $76\frac{1}{4}$‡, as the picture in the National Portrait Gallery in London. It is not only the age which associates the 1598 engraving with the London picture, but the lobe and arrangements of the left ear, cap and locks of hair are alike, and differ from that of the St Andrews portrait of this type, which indeed bears the age 75 and not 76 (see Plate X). Other pictures of the Bronckhorst type, such as those in the Edinburgh University Senate Hall (with scroll in right hand) and at Glasgow University need not be considered from this standpoint. One item may, however, be referred to. The Granthomme engraving in Boissard has a large fur collar. Such a collar it is difficult to discover in the London or St Andrews pictures, but it does appear in the second University of Edinburgh portrait—that with a book and hands resting on a balustrade. The lobe of the left ear and the hair arrangement are however

* Hume Brown: *John Knox*, 1895, Vol. II. pp. 322—324.

† No trust can be put in the Scottish spelling of foreign names in those days. This name may be identical with Vansome, in which form it also occurs, or with that of Vansomer. The best known painter of the latter name, as Mr H. M. Hake informs me, was active in the seventeenth century, but had progenitors of less importance in the sixteenth century.

‡ Not the age 73 of the Vansom portrait be it noted.

very different (cf. Plates III, IV, X, XI and XIII). It is very difficult to believe that the Edinburgh balustrade portrait is an independent portrait of Buchanan made two or three years before the Bronckhorst—the attitude is so completely the same, and the motto upon it is the same*. A copyist might omit the hanging lobe of the ear, but he would be extremely unlikely to introduce it *de novo*. The Edinburgh scroll picture, the Glasgow picture—which is almost certainly an early copy—and the St Andrews portrait of this type, which is dated a year before the London, leave the left ear in obscurity. On this account I am inclined to give precedence to the London portrait over all these with its marked individuality of left ear.

Another "balustrade" type of portraiture has come to my knowledge since my lecture; it is in the possession of Mr T. D. Findlay of Glasgow. It is difficult to say whether the Edinburgh or Glasgow portrait must be considered the original. The Glasgow portrait bears the same words "*Sic Buchananus ora*, etc.," but has no statement of Buchanan's age. The fact that the book and hands are thrown more into shade than in the Senate Hall picture, and a general impression of more life and less woodenness, are in favour of Mr Findlay's picture; but on the other hand the details of the hair over the left ear and of the fur collar are much more highly elaborated in the Edinburgh picture. I must leave my readers to judge for themselves which picture is the earlier. See Plates XI and XI *bis*.

Now it appears to be universally agreed by those who have written on the portraits that the London and the St Andrews pictures are by Bronckhorst, and that the Chouet engraving follows the missing portrait by Vansom sent to Beza, but if this be so, how does it come about that the Chouet engraving has for its *right* ear the peculiar lobe, the cap covering a part of the upper ear and the hair arrangement of the Bronckhorst *left* ear? We may suppose one to be a reversal of the other, made by the artist drawing directly onto the plate. But why should the cap be just covering the upper part of the ear and the curl be practically the same in two independent portraits, taken by different artists at an interval of two or three years? Further the London picture shows Buchanan with a black velvet cap, the Boissard-Granthomme engraving has this velvet, and the crumpling of the velvet is indicated by a series of "light" lines or patches descending to the forehead. These light effects are surely misinterpreted in the Chouet, and instead of a black velvet cap, when the wood-engraver sets to work, we have ruled lines, indicating a pattern on the cap, making it look like a printed cotton night-cap rather than the usual scholar's head-dress. No such cotton night-cap appears anywhere else in Beza's *Icones* or Chouet's engravings (see Plate III). I suggest that there is strong evidence that the Vansom painting of Buchanan never reached Beza or, if it did, he never had it engraved, but that Chouet, when 90 years later

* Mr William Carruthers (George Buchanan, *Glasgow Quatercentenary Studies*, 1906, Maclehose and Sons, 1907, p. 359) associated this portrait closely with the London one, without saying that it is a copy. He says that on the upper right corner is written "Ann. Aet. 76," i.e. the same date as the London picture, but my photograph of it has quite clearly, not on the upper right, but the upper left corner, "Aetatis 73," which if we could trust it, would make this picture three years older than the London picture and the subject's age the same as that of the Vansom picture.

he reproduced Beza's *Icones* with 40 new portraits, got a wood-cutter to copy in crude outline without reversing Granthomme's engraving of 1598. It is easy therefore to explain how "one man" has been said to be represented by the Vansom-Beza and the Bronckhorst-Boissard engravings. We do not get proof of the truthfulness of the Bronckhorst portrait if, as I think must be held, the Chouet is merely a crude copy of it. What also becomes of Sir William Hamilton's statement that the Chouet must represent the real man because it fits the skull better? It fits the skull better because the crude wood-cutter has altered the proportions of height to breadth of face. He has given the pear-shaped London face a more apple-shaped form, while preserving the disproportionately long nose, and (for the skull) impossible nasal bridge.

Those who hold that the Chouet woodcut was really prepared from the Vansom painting of 1579, when Buchanan was 73—and when considering this they must remember the Edinburgh balustrade picture, which is given as of age 73—must give up the idea that the London National Gallery picture is the Bronckhorst painting, notwithstanding its date, and suppose it also to belong to the Vansom group. I think they will find it easier to suppose that Chouet copied from Boissard.

We must now ask what is the evidence that associates the London and St Andrews portraits with Bronckhorst, and makes the Edinburgh and Glasgow pictures of this ancestry.

The evidence consists in Bronckhorst's bill to James VI, and his majesty's order for its payment, dated September 1580. The account runs: "Certane portraitures maid by me at his maiesties commaund and delyuerit laitlie to his hienes quherof I have resauit as yit na payement." The first is a portrait of "his maiestie" from the belt, and the price "xvj lib." The second item runs: "Ane other pourtraiet of Mr george buchanane. viii lib." (only half as much for a subject's portrait as for a king's!) and the third item a full length portrait of "his maiestie" xl lib., which shows the value of the royal legs.

The Buchanan item "Ane other pourtraiet" may mean that the first portrait was that of the King, but it might also mean that Bronckhorst had painted a previous portrait of Buchanan. Arnold van Brounckhurst* must have painted his picture or pictures before September 1580. The St Andrews portrait has on it the date 1580, and the age 75; the London portrait is dated 1581, with the age 76. Both portraits are clearly closely related to each other, the St Andrews being in a much worse state of preservation.

In 1580 James was only a boy 14 years of age, and he may or may not have wished for a portrait of his tutor. The order for payment is countersigned by Angus and Argyle, who then dominated the situation and may themselves have desired the portrait of Buchanan. What James thought of Buchanan and his works in his later years is well known, and in the lists of paintings belonging to Charles I prepared after his execution, no portrait of Buchanan appears in royal ownership.

* Thus the name seems to be spelt in the Scottish records. This artist was probably related to the better known and better spelt Bronckhorst of the seventeenth century.

Now without denying that the whole group of portraits to which we are referring may have originated in the Bronckhorst painting, there is still something obscure about them. In the first place there is the inscription on three of them, which runs: "Sic Bvchananvs ora, sic vvltvm tvlit. Pete scripta et astra, nosse si mentem cvpis."

Mr Carruthers translates this as follows: "This is the face and these are the features of Buchanan. If you want to know his mind, search his writings and the stars."

Now I have forgotten the little Latin I ever knew, but I should translate the sentences: "Thus Buchanan bore his features, thus his face. If you seek his spirit, search his writings and heaven."

In other words the verb is in the perfect and strongly suggests that Buchanan was dead at the time the writing was affixed to the picture. His "mens," the "divina mens," is to be found only in his writings and heaven. Further, Buchanan died on September 28, 1582, and is said to have been born in February 1506, or if that be Old Style, February 1507; he could therefore have only been 75 or at most 76 in the year of his death, 1582, while the London portrait makes him 76 in 1581 and the St Andrews portrait 75 in 1580. Thus the dates on these pictures appear to be erroneous, and I believe the dates and inscriptions must have been put on, and probably the pictures painted, after Buchanan's death. They may all have been copied from an original Bronckhorst which has not survived to our day. If they are copies and not only copies from the Bronckhorst, but one from another, we can easily explain the growth of idealised features and the emphasis of an intellectual spirituality, which may have been far less intensely present in the actual man himself.

In connection with this type of portrait two engravings occur, copies of which are in the Print Room of the British Museum (see Plate XII). The first of these is closely related to the Edinburgh "balustrade" picture. The portrait has an ornamental border by Karel van Sichem, and it is said to be painted by an artist whose monogram, there given, has not been identified. The engraving itself is by a second van Sichem, possibly Christoph, the son of the former. The original ornamental border enclosed an oval portrait which has been cleaned off and replaced by a square frame containing Buchanan. It has been suggested that the print was prepared in Cologne early in the 17th century. One would suspect that it was issued as a frontispiece to some edition of one or other of Buchanan's works, published in Cologne or the Netherlands, but I have not hitherto been able to identify it. It has the usual lines "*Sic Buchananus ora*, etc." A second print (see also Plate XII), obviously a copy of the van Sichem, was drawn by Elias Boschius (van der Bosche?) and engraved by Johan Bussemecher. This is still the Bronckhorst type, but a very inferior copy of the van Sichem. If the statement on the van Sichem plate be correct, the Edinburgh balustrade portrait, however much it resembles the Bronckhorst type, was not itself painted by Bronckhorst, but by the hitherto undetermined artist of the monogram.

We will shortly determine to what extent the skull fits the Bronckhorst group of portraits. Meanwhile we ask what other portraits or pseudo-portraits are available. A very useful graphic index to the various types of portraiture of Buchanan was provided by Mr Maitland Anderson in his pamphlet issued for the St Andrews Quatercentenary*, which he has kindly permitted me to reproduce. It shows four pictures of the Bronckhorst type, two of the Buchan type, and two of the Povey type. These will be found useful for showing at a glance the different types of portraiture with which we have to deal (see Plate XIV).

In the first place there is the Earl of Buchan's so-called "Titian" and its reproductions by Raeburn and others. This picture is at St Andrews, and is not to be confused with the St Andrews picture of the Bronckhorst type. The person represented is not the Buchanan of any other type of portraiture; the face does not fit the skull, and there is little doubt that it is a copy of the portrait of President Jeannin of Dijon which hangs in the Musée of that city (see Plate XV). We can therefore at once dismiss the "Titian" type as no portrait of Buchanan at all.

The next type requires far more consideration. We are certain of at least three portraits of this character, which I will call the "Royal Society or Povey type." Thomas Povey was a member of the Long Parliament in 1647, so it is reasonable to suppose that he was born between 1610 and 1620, i.e. from 30 to 40 years after the death of Buchanan. He was a friend of Pepys and an original member of the Royal Society from 1663. To that Society he presented a portrait of George Buchanan, which has painted on it: Georgivs Bvchanan Scotvs (see Plate XVI).

The costume is that of Buchanan's date. The face is wholly unlike that of the Bronckhorst type. Now the portrait has been some 260 years in the possession of the Royal Society, and therefore the ascription to Buchanan is not a recent one. It carries us back to a date, when probably some of those living had seen Buchanan. Povey was a favourite at Court, and Charles II, the grandson of Buchanan's pupil, who frequented the meetings of the Royal Society, could hardly fail to have seen portraits of his grandfather's tutor. Mr Carruthers, who dismisses this portrait, says that George Buchanan was a common name in Scotland; that two George Buchanans might have had their portraits painted, and that the inscription only says "George Buchanan, *a* Scot." I think it would be more reasonable to translate it: "George Buchanan, *the* Scot," and I think there was only one man of that name who reached fame at that date. Now it is quite clear that the word *Scotus* is significant. If this George Buchanan is not the real man, but some rich merchant who visited the Netherlands and had his portrait painted there, as Mr Carruthers suggests, why was *Scotus* put on the picture? It was hardly needful to identify him in the family portrait gallery! Is it not probable that this portrait, whoever it may be of, was being sent abroad—to England, to Denmark, or even to Geneva,—in which case the *Scotus* was appropriate?

Further, in all the editions of Buchanan's works issued in his life, the author on the title page appears as "Georgius Buchanan Scotus," which is certainly "George

* *The Writings and Portraits of George Buchanan*, University Press, St Andrews, 1906.

Buchanan, *the* Scot." But Povey's George Buchanan does not stand as a sole representative of this type. Dr Richard Mead, the celebrated physician, born in 1673 (d. 1754) was a man of wide knowledge, who had studied under Graevius in Utrecht and Boerhaave in Leyden. He had travelled in Italy and in other parts of Europe and amassed a great library, a gallery of well-chosen pictures and a collection of coins and other antiquities. His library alone contained 10,000 volumes and his pictures, books, etc. sold for more than £16,000 on his death—a very large sum for those days. He built for his pictures and antiquities a gallery attached to his house in Great Ormond Street, London. Among his pictures was a portrait of George Buchanan. This painting is almost certainly the one now in possession of the Duke of Sutherland. It was engraved by Houbraken in 1741 and the picture was then attributed to Porbus (see Plates XVII and XVIII). I do not think this portrait was a copy made for Dr Mead of the Royal Society picture, but it is undoubtedly of the same man and must, judging from the hair, have been taken at a somewhat later time. Why should the supposititious rich merchant of Edinburgh have had no less than three paintings of himself? How did two of them come to London to fall into the hands of two fellows of the Royal Society?—I think these portraits tend mutually to strengthen each other, and to give the Royal Society type a strong claim to be considered when we compare skull and portraits.

Still further confirmation of the originality of this type of portrait has turned up since this lecture was given in Edinburgh, namely a picture in the possession of Mr George K. Sowersby, which is also a portrait of early date with the same expression of features but without any title (see Plate XIX). Mr Carruthers' worthy merchant of Edinburgh must indeed have been fond of having paintings made of himself. It is difficult to place exactly Mr Sowersby's portrait. It is certainly not a copy of the Royal Society portrait, being much closer to the Dunrobin picture. There is no doubt that it is the same man, but it would be unwise to assert at present which is the original.

Like Dr Mead's portrait the Royal Society's has been attributed to Porbus, a Flemish artist, but more recently to Adrian Keij. It is clearly Flemish work. There is no evidence that Porbus, either father or son, came to Scotland, and little certain evidence that Buchanan visited the Netherlands. But the difficulty about a Flemish painter is not really great, as Flemish painters were by no means unknown in Edinburgh. Now we know that Tycho Brahé in his Observatory at the Uranienberg fitted up for him by Fredrik II of Denmark had a portrait of George Buchanan. Brahé and Buchanan had been in correspondence in 1576. Tycho Brahé travelled in Germany and Switzerland in 1575, and possibly then made the acquaintance of Beza and through him of Buchanan. It must also be remembered that from 1560 onwards Buchanan had been busy with his poem *De Sphæra*, which takes the anti-Copernican view—what we might in modern language term the "Fundamentalist" view—of the universe. This was also Brahé's standpoint, and Buchanan may well have sought his advice and interpretation against the Copernican views of the more advanced school of heretics. Buchanan's pupil James VI recognised this portrait of his old tutor, when on going

to fetch his wife Anne of Denmark he visited Tycho Brahé at the Uranienberg. Brahé left Denmark in 1597 and died in Prag in 1601. Buchanan's portrait has been sought in vain in Denmark and Bohemia. If it was left at the Uranienberg it might well pass back to the Danish Royal Family with the observatory, and may have been sent to Queen Anne as of more interest to her husband than to the Danes. But what has this to do with Povey? Just this, Povey was son of Justinian Povey who was treasurer to Anne of Denmark, that frivolous but kindly-hearted queen, who never could keep her expenditure within her income, and was continually pledging her jewels and valuables even to her treasurer. If Buchanan's portrait came from the Uranienberg to Queen Anne it may very possibly have passed to her treasurer Justinian Povey, and thence to his son Thomas who ultimately gave it to the Royal Society*. All these stages are only conjectures, but not very improbable conjectures. It has been suggested that Peter Young, who was in Denmark in 1585 and 1589 about the marriage of James to Anne, presented a portrait of Buchanan to Tycho Brahé, but it is not very clear why he should have done so either 3 or 7 years after Buchanan's death. It seems more reasonable to suppose that Buchanan himself sent his portrait to Brahé. The picture, while it may represent a man well on in the sixties, is more appropriate to that age than to the seventies.

Whence Dr Mead obtained his portrait it is difficult to suggest; if as seems probable, it is the one now in the possession of the Duke of Sutherland at Dunrobin Castle, then the picture is of the same man as the Royal Society portrait, but as I have said he is several years older, for the hair of the head, moustache and beard are all grey. Is it conceivable that Dr Mead's portrait was the Vansom sent to Beza? If that were so, then the Tycho Brahé and the Beza portraits need no longer be treated as missing, and we should have a second type of Buchanan very different from the Bronckhorst type.

Mr Carruthers speaks of the Povey-Mead type of Buchanan portrait as if the tradition of this portrait type had started with some error of assignment by Povey in the last quarter of the 17th century, and that from Povey's error had arisen the whole tradition of this portrait type representing Buchanan. Mr Carruthers seems to have entirely overlooked the facts that: (i) the Povey and Mead portraits represent the same man at *different* ages and (ii)—what is far more important—the so-called Povey type is the type which appears in the Elzevir edition of the *Poemata* in 1628, which is obviously engraved from a picture of the Povey type, if not from the Povey picture itself. In 1628—James I only died in 1625—there must have been persons still living who had known Buchanan in the flesh, yet this passed as Buchanan's likeness not only in this edition of his poems, but in those issued at Amsterdam in 1676 and 1687, until it appears in a highly idealised form in Ruddiman's great folio edition of the *Opera Omnia* published in Edinburgh

* An absurd statement occurs on p. 534 of the *Glasgow Quatercentenary Studies* (George Buchanan), 1906, where we are told Povey presented the picture to the Royal Society in 1633, i.e. about 30 years before the foundation of that Society and when Povey was a very young man just beginning to study for the Bar! He lived on into James II's reign, and probably died towards the end of the century.

in 1715 (see Plates XX and XXI). In other words the Royal Society type was as well known and as often—perhaps more often—published in the century which followed Buchanan's death as the Bronckhorst type. Despite Mr Carruthers, traditional evidence for the Povey type is as strong as for the Boissard-Bronckhorst type. It is not until Burmann's re-issue of Ruddiman in 1725 that the Boissard-Bronckhorst type replaces the Povey type in the *Opera* (see Plate XXII).

It has been said that there was no opportunity for a painting being made of Buchanan when he was about 60 years of age. This overlooks the fact that in 1565—6 Buchanan visited Paris to see his Latin paraphrase of the Psalms passed through the press by Henri Estienne and his brother *. On the title page of that work Buchanan appears as Georgius Buchananus Scotus† "poetarum nostri saeculi facilé princeps." A man with such a reputation might well have his portrait painted. How Buchanan travelled to Paris we do not know, very possibly through the Netherlands, and there is a remarkable letter of Languet to Buchanan stating that he lives in sight of Rotterdam which reminds him not only of Erasmus but also of Buchanan. This suggests that he had met Buchanan in Rotterdam.

While the Bronckhorst and Royal Society types show Buchanan as "aged" if not "agéd," there exist two other portrait types quite incompatible with the Bronckhorst type (but more or less compatible with the Royal Society type) which represent, or profess to represent, Buchanan aged about 30 and 50 respectively. One of the former type was originally at Hamilton Palace, and from it Harding made an engraving for Pinkerton's "Scottish Gallery" (see Plate XXIII). By courtesy of the Duke of Hamilton, I have been informed: (1) by the Hamilton Estates Trustees that no such portrait now exists in Hamilton possession, and (2) by Messrs Christie, Manson & Woods, that no such portrait was sold with the Hamilton Palace collections. A portrait of this type on canvas has, however, turned up in the possession of Mr H. B. Buchanan, which is undoubtedly an old painting. It has been in the possession of his family for many years and may possibly have reached them by purchase or gift from Hamilton Palace. I am very grateful for the permission to reproduce it in this paper. A second, somewhat idealised, version of this portrait is now in the Technical College at Glasgow, which again may be the original Hamilton Palace picture (see Plates XXIII, Right, XXIV and XXV).

One of the second type of portrait—i.e. of Buchanan about 50 years of age—is in the possession of Miss Christie of Bedlay. By her permission and that of the publishers of the *Glasgow Quatercentenary Studies* (George Buchanan), I have been allowed to reproduce it (see Plate XXIII, Left).

While the skull of an adult changes to some slight extent with age the amount is so small as to be quite insignificant compared with the personal equation of an

* Ruddiman in his *Anticrisis*, 1754, writes: "I have heard it related a hundred times that Buchanan when Principal of St Leonard's College, at St Andrews, did make such a voyage to France."

† It must be emphasised that the word *Scotus* almost invariably follows George Buchanan's name in any of his works published in his lifetime, and is thus in keeping with the title on the Royal Society portrait.

artist. His deviations from truth, even in the case of the most conscientious of the craft, are of quite a different order of magnitude. We can have no hesitation then in applying the Edinburgh skull to the portraits reputed to be those of Buchanan in the years from 30 to 50.

I will now run rapidly through the portraits. I have to thank Miss Ida McLearn for measured drawings of the portraits to each of which we have attached outlines of photographs of the skull in the corresponding attitude.

(1) *Portrait of a Young Man. Glasgow Technical College Portrait.*

With a slight allowance for artistic inaccuracy the skull might almost fit this portrait. We cannot, however, raise the alveolar margin adequately without getting the orbits and nasal bridge out of position. The frontals of the skull are scarcely high enough for the forehead of the portrait (see Plate XXVI, Left).

Allowing for artistic inaccuracies I do not think we must venture to say on the authority of the skull that this type of portrait does *not* represent Buchanan when young.

(2) *Miss Christie's Portrait of a Man between* 40 *and* 50.

Again this is not such a *very* bad fit. Forehead too high and broad. Cheek bones about right and mastoid fairly good. Alveolar border too low and cannot be raised without sending orbits out of position (see Plate XXVI, Right).

It cannot be affirmed on the evidence of the skull that this is certainly not Buchanan.

(3) *Chouet* (*?Vansom*). Right (?left *reversed*) mastoid out of position, left cheek bone projects beyond portrait without flesh. Alveolar margin is under nose instead of under upper lip. Excess of right frontal. Would be a very bad portrait of the owner of the skull, if portrait at all (see Plate XXVII, Left).

(4) *George Buchanan from Boissard's Icones*, 1598. Left mastoid out of position, right cheek bone protruding beyond portrait without flesh. Alveolar margin of upper jaw under nose instead of under upper lip. Excess of right frontal. All the defects of the Chouet portrait, of course reversed (see Plate XXVII, Right).

(5) *George Buchanan, National Portrait Gallery.* This may stand as representative of the two Edinburgh, the St Andrews, the two Aberdeen and the Glasgow portraits. They are all of the same, the Bronckhorst, type (see Plate XXVIII).

Left mastoid out of place, nose much too long, so that it reaches to alveolar margin. Right cheek bone without flesh outside limits of portrait. Right frontal in excess. It is impossible to defend this portrait as a truthful portrait of the owner of the skull. Those who accept this portrait as a representation of George Buchanan must either discard the skull, or be willing to admit and approve extreme artistic licence.

(6) *Earl of Buchanan's "Titian," St Andrews* (*Copies at the Buchan Club, etc.*). I do not know whether it is worth the energy it has taken to dismiss this portrait

of President Jeannin, but I have done so. The orbits are out of position, the alveolar margin absurd, and the forehead ridiculous for the skull (see Plate XXIX, Left).

(7) *The Royal Society's Portrait.* The orbits are reasonable, so is the nose. The alveolar margin is in the true position; the cheek bones practically where they should be, and the frontals excellent. If the skull is to be our guide, this could hardly be bettered (see Plate XXX). On Plate XXIX, Right, will be found another illustration of this type, namely Mr Sowersby's portrait fitted with the skull. As in the case of the Royal Society picture the skull fits well.

We are, I think, driven to the conclusion that the skull accords with the Povey-Mead type of portrait, the type of which our earliest trace of authenticated date is the Elzevir *Poemata* of 1628.

Now what exactly does this mean for us as phrenologists? Let us look on the two portraits once more.

Let us take the highly idealised Bronckhorst portrait in Edinburgh as one type (see Plate XIII). Note the high forehead of great breadth, the marked bridge and long nose, the projecting cheek bones and stern look. We have here all the signs of the earnest religious reformer, the ascetic scholar and the stringent moralist; Buchanan as a second Cato.

Now look at the Royal Society picture (see Plate XVI). We see the exaggerated right orbit, the Socratic nose, and all the rotundity which identify man and skull. The portrait suggests the intimate of Muretus, the man who preferred a pawky story to true history, the man who could write poems worthy of Catullus; the man who could appeal alike to the surpassing talents of Henry VIII and of the Regent Moray, or sing the virtues of Mary Stewart and of Elizabeth Tudor, according as the moment was propitious for receiving their aid. The two portraits express the two attitudes that historians have taken to Buchanan, and whether we like it or not, each one of us has got to take his choice between Cato and Catullus. For as Buchanan's friend Muretus laid it down:

> "Quisquis versibus exprimit Catullum,
> Raro moribus exprimit Catonem."

The man who in his verses propounds Catullus, will scarcely exemplify Cato in his morality.

Here is a choice between alternatives*. Either you must reject the Cato portrait as no truthful likeness, or if you accept the Cato portrait as a *truthful* representation, you must return the skull to the Greyfriars as not that of George Buchanan, although it fits excellently that other portrait of "Georgivs Bvchanan Scotvs." The alternative lies in how you translate "Scotvs."

* The Chairman, Sir E. A. Schäffer, suggested at the end of the lecture that the two portraiture alternatives were those of Don Quixote and Sancho Panza. This somewhat apt description of the Edinburgh "Scroll" and the Royal Society pictures has really no application to the alternative interpretations of the indisputable mental ability of George Buchanan.

Jeremy Bentham,
with contour lines from a photograph of the "Auto-Icon."

Sir Thomas Browne
NATIONAL PORTRAIT GALLERY

Allix Portrait of Sir Thomas Browne (now in the National Portrait Gallery). The fit of the skull shows a passable portrait.

Sir Thomas Browne
ROYAL COLLEGE OF PHYSICIANS

Portrait of Sir Thomas Browne (in the Royal College of Physicians). The artist has idealised absurdly the forehead to emphasize intellectuality.

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The Granthomme Engraving from the Boissard, 1598.

The Chouet Woodcut, 1673.

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Bronckhorst type, from the National Portrait Gallery, London. Asserted date, 1581.
[*Reproduced by permission.*]

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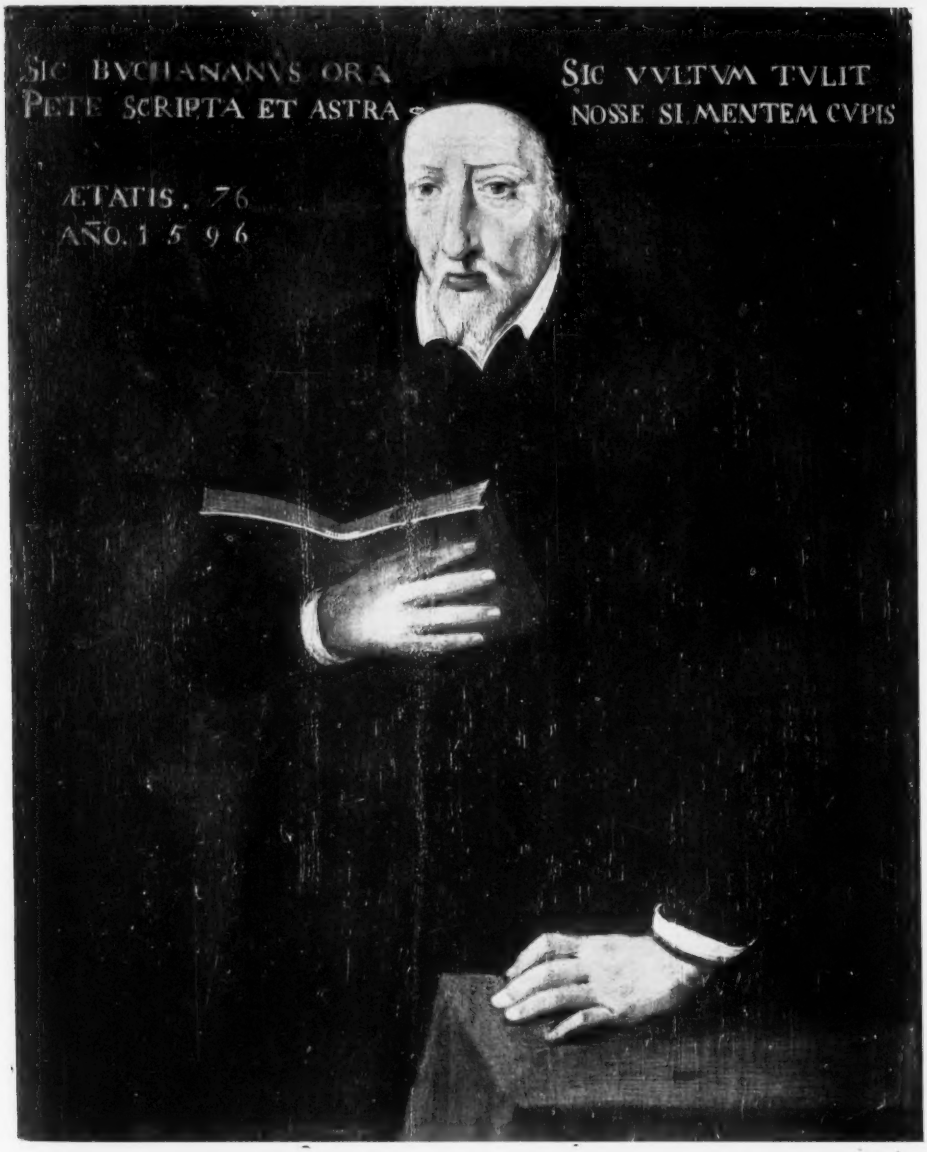

Bronckhorst type, from the Landesbibliothek, Wolfenbüttel. A copy made 14 years after Buchanan's death either from the National Portrait Gallery (London) painting, or more probably from a missing picture from which the latter also was copied. [*By kind permission.*]

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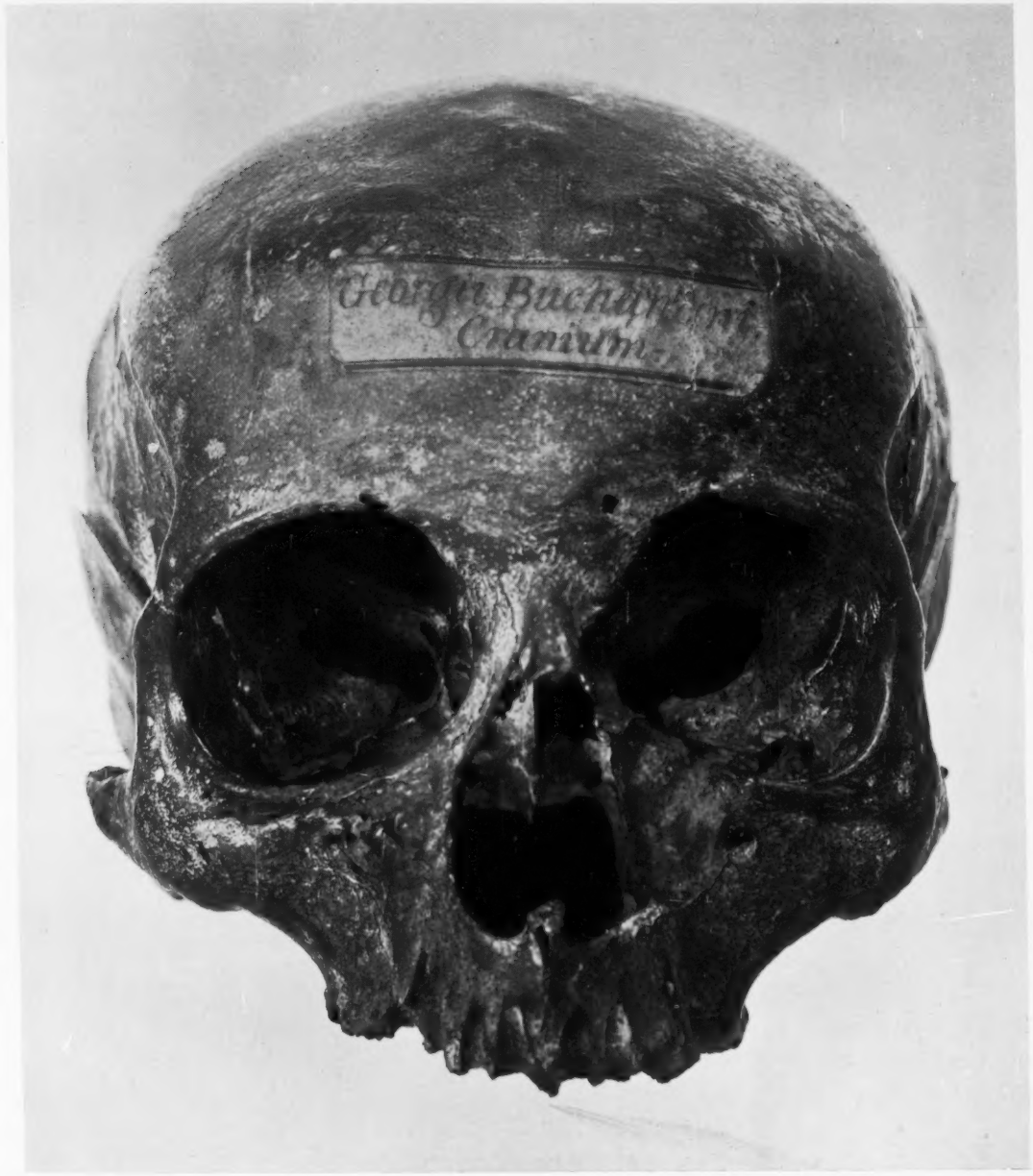

Buchanan's Skull; *Norma Facialis.*

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Buchanan's Skull; *Norma Verticalis.*

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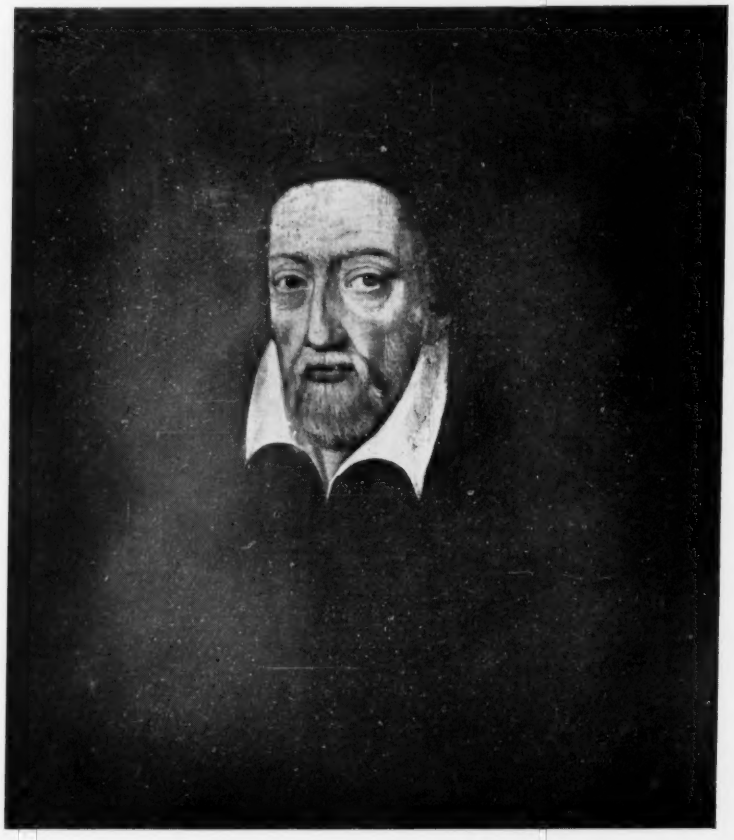
Buchanan's Skull; *Norma Lateralis* (Right Profile).

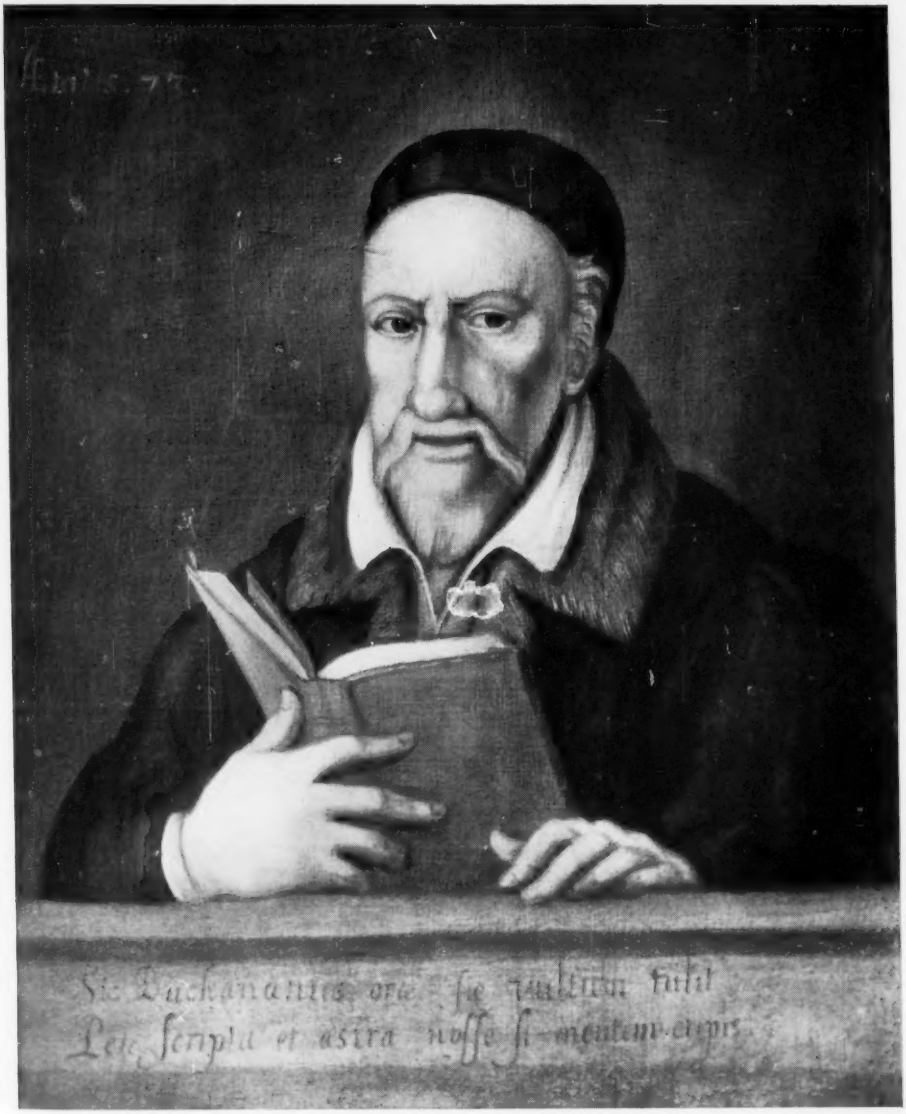
Buchanan's Skull; *Norma Lateralis* (Left Profile).

Buchanan's Skull; *Norma Occipitalis.*

Bronckhorst type, from St Andrews University. Asserted date, 1580. [*By kind permission.*]



Bronckhorst type, from Glasgow University. A copy of one of the earlier pictures. [*By kind permission.*]



"Balustrade" Portrait, from the Senate Hall, Edinburgh University. Asserted date, 1579. Clearly Bronckhorst type. [*By kind permission.*]

UNIV. OF MICH.

Portrait of George Buchanan of the Edinburgh "Balustrade" type in the possession of Mr T. D. Findlay of Glasgow. [*By kind permission.*]

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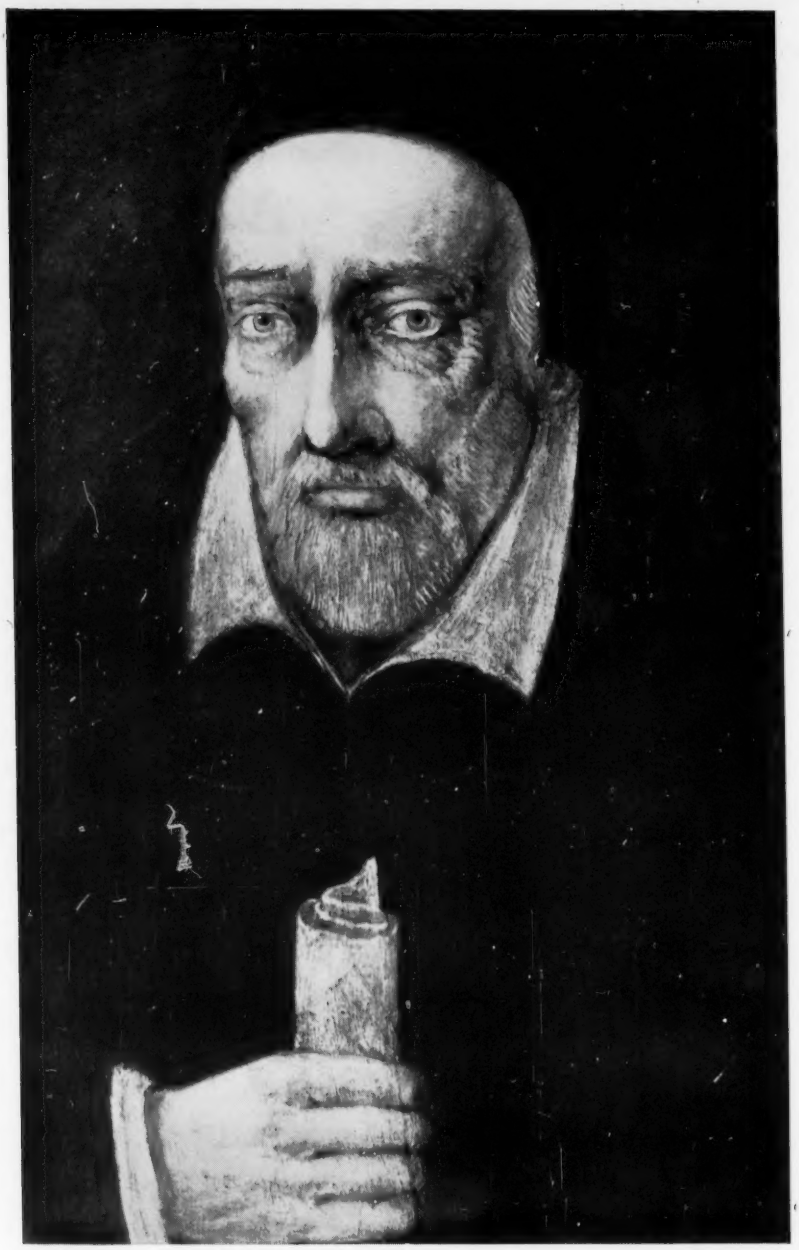
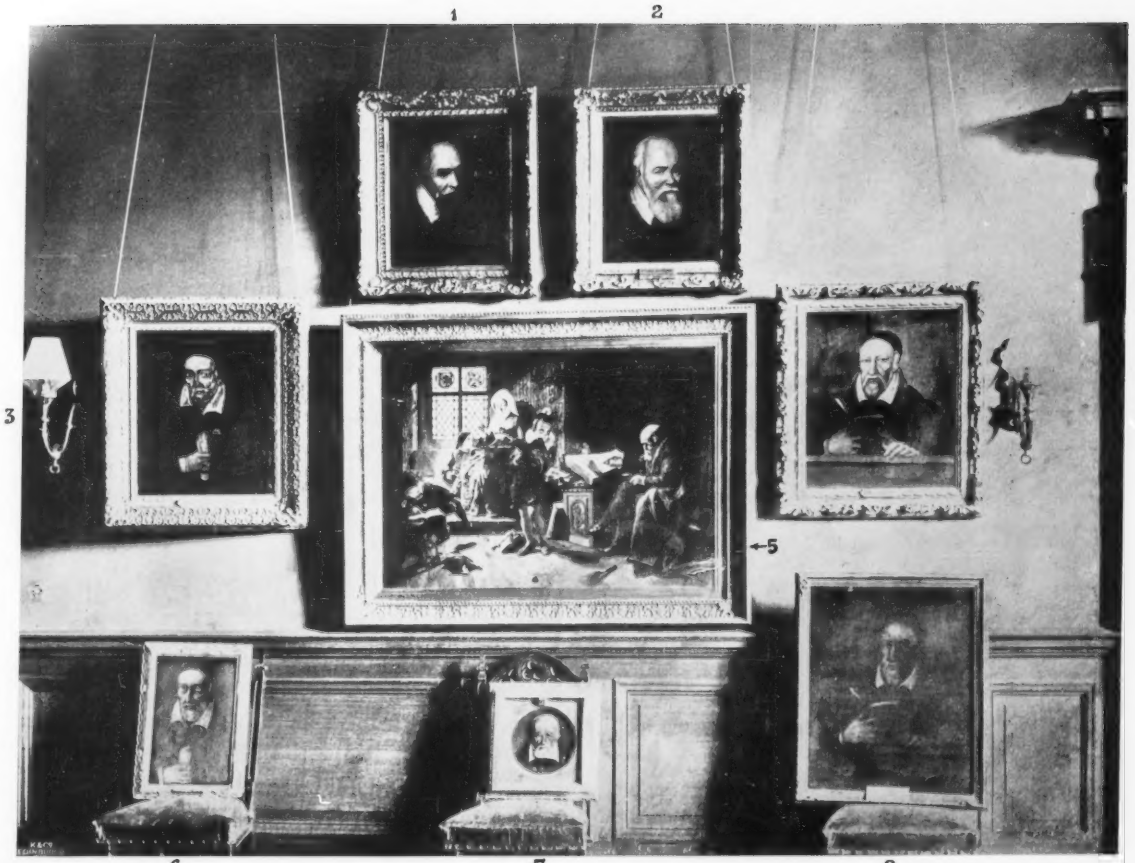
GEORGIVS BVCHANANVS SCOTVS
Hic est quem noui nec vidi; scilicet hic est
Ille Poëtarum qui Coryphæus erat.
Hic nos Diuinis rapiat super æthera Psalmis:
Priscaque Scotorum tot Monumenta docet.
Pulpita nunc celso celsus quatit ipse Cothurno;
Ardua nunc facilem sternit ad Astra viam.
Ecquid erat præmi? Coeli, quod carmine pandit
Doctiloquo, maneat incola perpetuus.
Johan Blaeuw excudit.
Elias Bosch sculpsit.

Engravings clearly made from the Edinburgh " Balustrade " type of portrait. From originals in the Department of Prints, British Museum. Probably used as title-pages for hitherto undetermined editions of some work or works of Buchanan. [See p. 250 of the text.] Probable dates first to second quarter of 17th century.

U. OF MICH.

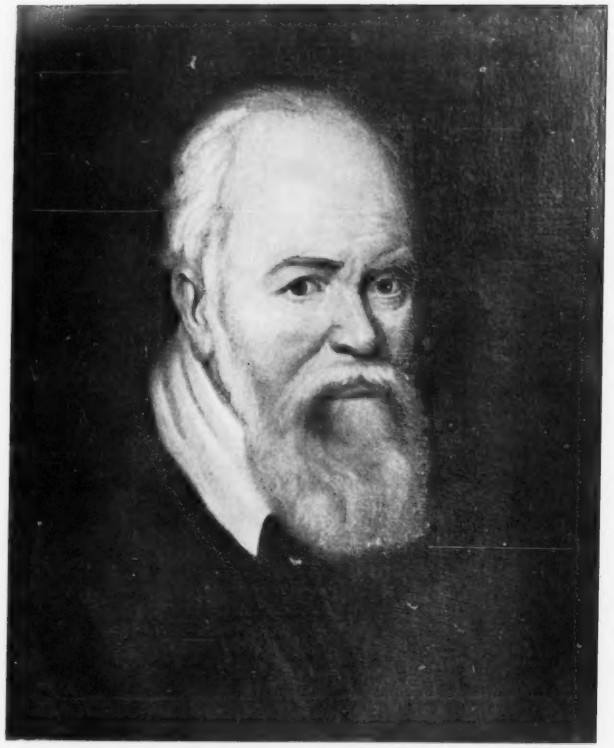


Highly idealised portrait of the Bronckhorst type, in the Senate Hall, Edinburgh University. Note the disappearance of the lobe of the left ear. Probably a rather late copy. [*By kind permission.*]

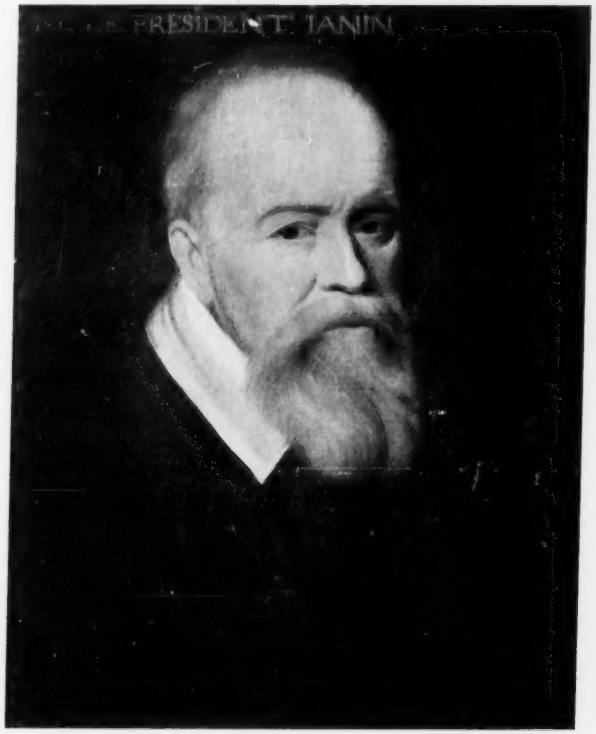


(1) Copy of 2 belonging to Buchan Society. (2) So called "Titian" portrait at St Andrews.
(3 & 4) The Aberdeen portraits, copies of the Edinburgh. (5) Mrs Buchanan's modern picture.
(6) Edinburgh. (7) Duke of Sutherland's. (8) Edinburgh.

General indication of the various types of Buchanan portraiture, taken by kind permission of Mr Maitland Anderson from his *The Writings and Portraits of George Buchanan*, 1906.

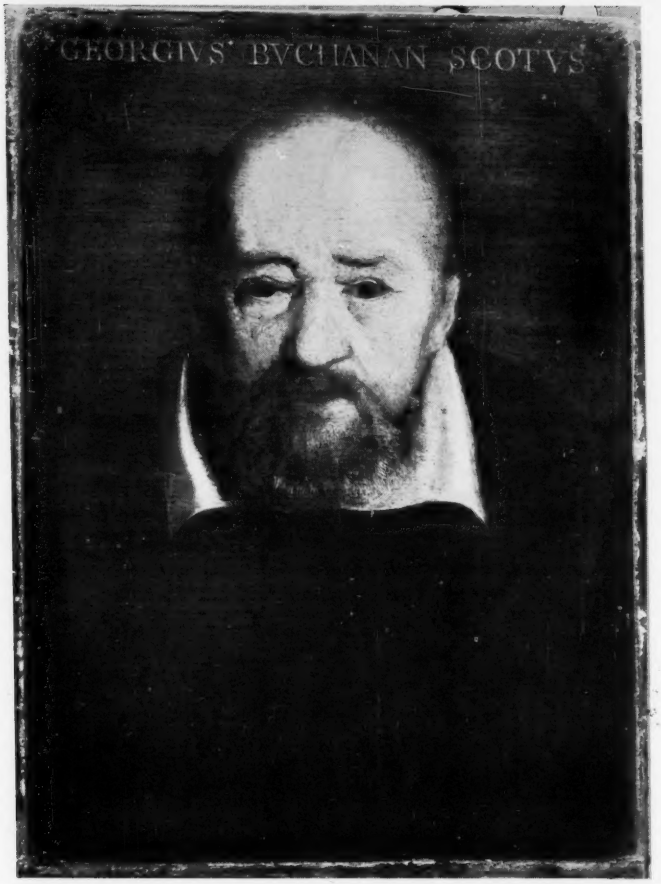


The so-called "Titian" portrait at St Andrews University, probably originally in the possession of the Earl of Buchan.
[*By kind permission.*]



Photograph of portrait of President Jeannin of Dijon, from the Musée of that city, showing that Titian's "Buchanan" is a faked portrait. [Photographed for this work.]

U. OF MICH.



The Royal Society or "Poemata" type of Buchanan portrait, originally attributed to Porbus, possibly by Adrian Keij.
[*By kind permission.*]

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The Dunrobin Portrait of Buchanan of the "Poemata" type, formerly in the possession of Dr Mead, and now in the possession of the Duke of Sutherland.

[*By kind permission.*]

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The Houbraken engraving (reduced) of Dr Mead's portrait which identifies the latter with the picture now at Dunrobin Castle.

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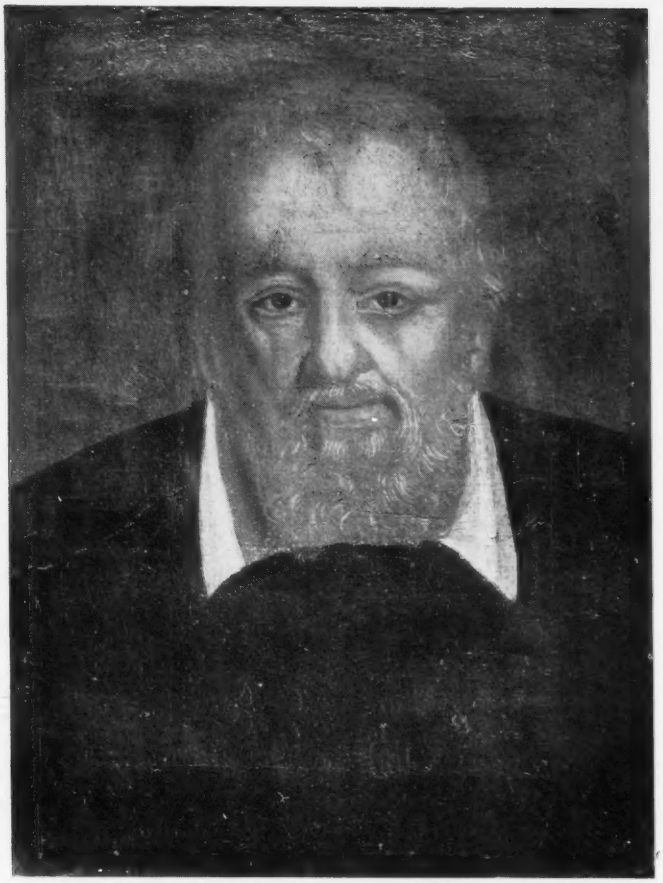
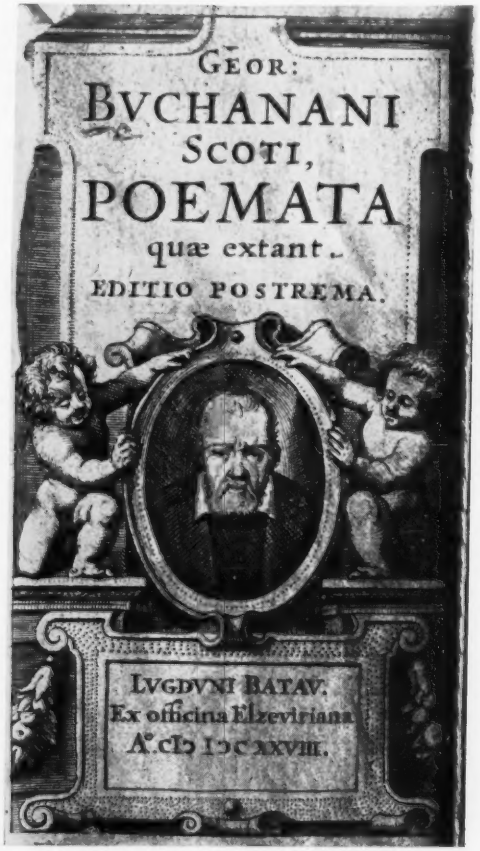


Mr Sowersby's Portrait of Buchanan of the "Poemata" type, said to have been formerly in the possession of the Dowager Countess of Seafield. [*By kind permission.*]



The "Poemata" type of Buchanan portrait, in agreement with the Povey, Mead and Sowersby Portraits.
First issued in 1628 and maintained in the Amsterdam editions of the 17th century.

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Title-page of the first Edinburgh Edition of the *Opera Omnia*, by Ruddiman in 1715, showing a highly idealised medallion of Buchanan of the "Poemata" type of portraiture. [Reproduced, much reduced, from a copy in the possession of Dr James Bonar.]

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ÆTATIS LXXVI
GEORGIUS
BUCHANANUS
POESEOS
Historia
Scotica
&c.
Hic est, quem nôsse, nemo non, scilicet hic est
Miles Poetarum, Musicus, eruditus.
Hic Buchananus
Tam bene Sichem
Salve, Mores, salve Leges procul este Profani.
Salve. Horos, salve, quidquid Frater-rimus Ordo
Grummiat: o salve nos tua turba sumus.
Nos tibi devovet, nos nos tibi ponimus Aras.
Tu nos devinxisti raptus super æthera Psalmis.
Perfecit, Sacrorum tot Monumenta decus.
Nunc tua facilem sternus in Astra viam.
Et quod erat præmi Cælum, quem carmine pangis.
Hunc tibi quid totâ mente vovemus habe.
Petrus Scrivenius.

Re-introduction of the Bronckhorst type in the re-issue by Burmann of the *Opera Omnia*, 1725. [Reduced from quarto format.]

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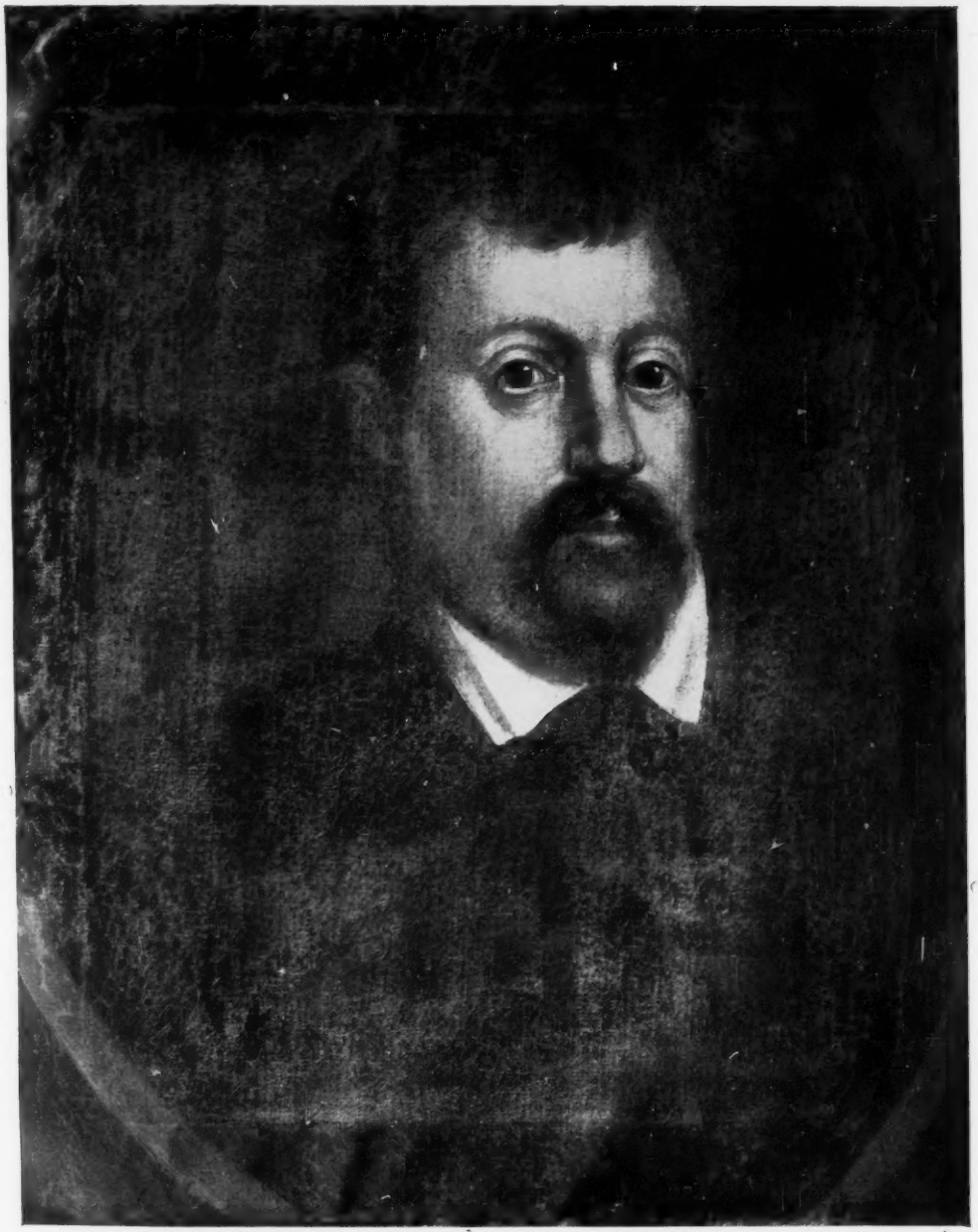
Miss Christie's Portrait, said to be the middle-aged Buchanan. [*By kind permission of the owner and of the publishers of the* "Glasgow Quatercentenary Studies."]

The Harding engraving of the youthful Buchanan stated at the time when it was engraved to be at Hamilton Palace. [See p. 254 of the text.]

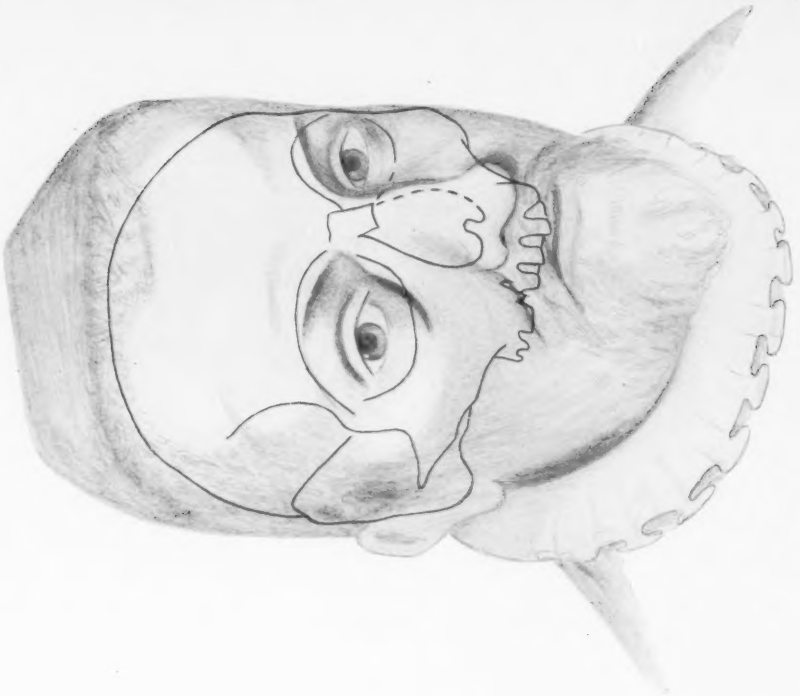


Mr H. B. Buchanan's picture of the youthful Buchanan. This seems to be a more original and less idealised portrait than that of the Glasgow Technical College.

[*By kind permission.*]

UNIV. OF MICH.

Portrait of the youthful Buchanan in the Technical College, Glasgow. Probably a copy of that originally at Hamilton Palace. [*By kind permission.*]



George Buchanan
In Possession of Miss Christie

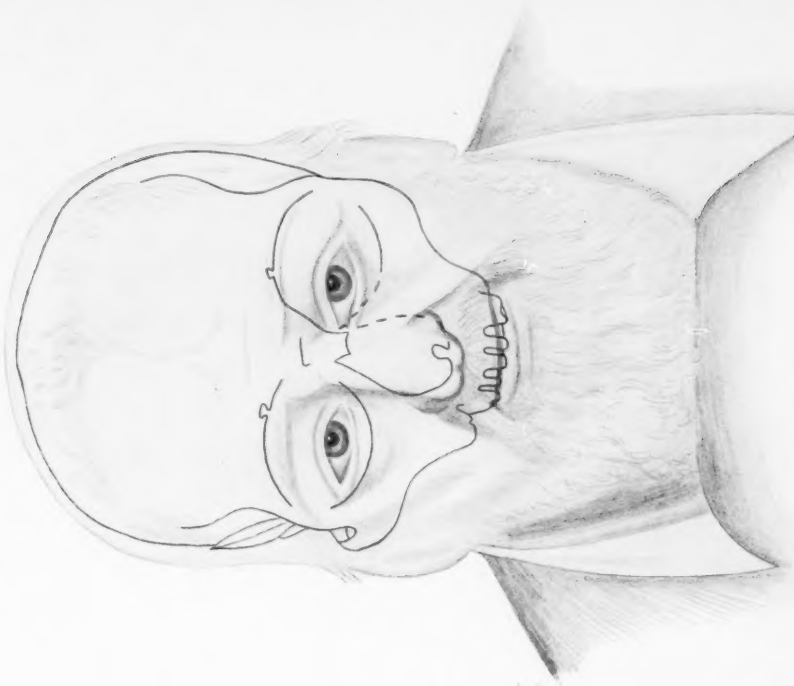
George Buchanan
Technical College, Glasgow

George Buchanan
Boissard's Icones

George Buchanan
Supposed Vanson
Chouet Woodcut.

UNIV. OF MICH.

George Buchanan
NATIONAL PORTRAIT GALLERY



George Buchanan
Belonging to George K. Sowerby, Esq.
Formerly in the possession of the Dowager Countess of Seafield

George Buchanan
St Andrews "Titian"

UNIV. OF MICH.

George Buchanan
ROYAL SOCIETY OF LONDON

UNIV OF MICH

FURTHER NOTES ON NON-LINEAR REGRESSION.

By J. NEYMAN, Ph.D., Warsaw.

1. The theory of non-linear regression has been discussed by Prof. K. Pearson in his memoir *On the General Theory of Skew Correlation and Non-Linear Regression* (London, 1905), and in a paper in *Biometrika*, Vol. XIII. pp. 296—300. In these two papers the general method of calculation of polynomials, which are approximations to the unknown skew regression line, is given and formulae are deduced expressing the coefficients of such polynomials as far as the fourth order in terms of the frequency constants of the population considered.

The results of Heine* combined with those of H. Hamburger†, who worked with the theory of continued fractions, allow us to put the results of K. Pearson into a more general form.

We shall do it, however, without any appeal to the theory of continued fractions.

In a further paper we shall apply these results to the correlation between the standard deviation squared and the mean in a sample drawn from an indefinitely large population.

Higher Parabolae of Regression and Higher Correlation.

2. Let x and y be two variates, and $z = f(x, y)$ the function representing their frequency distribution. The two variates may have their possible values forming a continuous set of points (as it is, e.g., in the case when their distribution is normal), or not (as it is in the case when the distribution is, e.g., binomial). We shall use nevertheless the notation of $\int$, which in the last case is to be considered as the integral of Stieltjes, i.e. as a sum.

The integrals over the entire range of each of the two variates we shall denote by $\int_{-\infty}^{+\infty}$; as in the case when the range is limited, it is always possible to suppose that outside this range

$$f(x, y) = 0 \quad \ldots\ldots\ldots\

Let $\bar{y}_x$ be the mean value of y corresponding to a given value of x. We have

$$\bar{y}_x = \frac{\int_{-\infty}^{+\infty} yf(x,y)\,dy}{\int_{-\infty}^{+\infty} f(x,y)\,dy} = \frac{\phi(x)}{p(x)} \quad \ldots\ldots(2),$$

if

$$\int_{-\infty}^{+\infty} yf(x,y)\,dy = \phi(x) \quad \ldots\ldots(3)$$

and

$$\int_{-\infty}^{+\infty} f(x,y)\,dy = p(x) \quad \ldots\ldots(4).$$

In trying to approximate to the generally unknown function $\frac{\phi(x)}{p(x)}$, we shall calculate a polynomial of the n-th degree

$$\tilde{y} = P_n = a_0 + a_1 x + a_2 x^2 + \ldots + a_k x^k + \ldots + a_n x^n \quad \ldots\ldots(5),$$

satisfying the condition, that the integral

$$I = \int_{-\infty}^{+\infty} \left[\frac{\phi(x)}{p(x)} - P_n\right]^2 p(x)\,dx \quad \ldots\ldots(6)$$

is a minimum. If we observe, that $p(x)\,dx$ is nothing but the probability that the variate x should lie in the subrange dx, the meaning of this integral is clear.

As I is always a differentiable function of the a's, we use the ordinary method for finding minima, and solve the system of $n+1$ linear equations in the a's, given by equating to zero the consecutive derivates $\frac{\partial I}{\partial a_k}(k = 0, 1, 2 \ldots n)$, namely,

$$a_0\mu_k + a_1\mu_{k+1} + a_2\mu_{k+2} + \ldots + a_k\mu_{2k} + \ldots + a_n\mu_{k+n} = \lambda_k \quad \ldots\ldots(7),$$

where $\mu_k = \int_{-\infty}^{+\infty} x^k p(x)\,dx$, $\lambda_k = \int_{-\infty}^{+\infty} x^k \phi(x)\,dx = \int_{\infty}^{+\infty}\int_{-\infty}^{+\infty} yx^k f(x,y)\,dx\,dy \ldots(8)$.

If the origin of the coordinates be at the mean of the two variates, we have $\mu_1 = \lambda_0 = 0$. We notice also that $\mu_0 = 1$.

As the determinant of the system of equations (7)

$$\Delta_n = \begin{vmatrix} \mu_0, & \mu_1, & \ldots & \mu_k, & \ldots & \mu_n \\ \mu_1, & \mu_2, & \ldots & \mu_{k+1}, & \ldots & \mu_{n+1} \\ \ldots\ldots & & & & & \\ \mu_k, & \mu_{k+1}, & \ldots & \mu_{2k}, & \ldots & \mu_{k+n} \\ \ldots\ldots & & & & & \\ \mu_n, & \mu_{n+1}, & \ldots & \mu_{n+k}, & \ldots & \mu_{2n} \end{vmatrix} \quad \ldots\ldots(9)$$

is always >0*, system (9) gives solutions for the a's which are not all equal to zero when, and only when, the λ_k's are not all equal to zero. Such zero solutions have, however, no interest for us. Considering the system of $n+2$ equations

$$\left.\begin{aligned} -\tilde{y} + a_0 + a_1 x + a_2 x^2 + \ldots + a_k x^k + \ldots + a_n x^n = 0 \\ \text{and} \quad -\lambda_k + a_0\mu_k + a_1\mu_{k+1} + a_2\mu_{k+2} + \ldots + a_k\mu_{2k} + \ldots + a_n\mu_{k+n} = 0 \\ (k = 0, 1, 2 \ldots n) \end{aligned}\right\} \quad \ldots(10),$$

* T. J. Stieltjes, "Recherches sur les fractions continues," *Ann. de la fac. des sc. de Toulouse*, 1894.

we can free ourselves of the a's and write the necessary and sufficient conditions that they should be satisfied as

$$\begin{vmatrix} -\tilde{y}, & 1, & x, & x^2, & \dots & x^k, & \dots & x^n \\ -\lambda_0, & \mu_0, & \mu_1, & \mu_2, & \dots & \mu_k, & \dots & \mu_n \\ -\lambda_1, & \mu_1, & \mu_2, & \mu_3, & \dots & \mu_{k+1}, & \dots & \mu_{n+1} \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ -\lambda_k, & \mu_k, & \mu_{k+1}, & \mu_{k+2}, & \dots & \mu_{2k}, & \dots & \mu_{k+n} \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ -\lambda_n, & \mu_n, & \mu_{n+1}, & \mu_{n+2}, & \dots & \mu_{n+k}, & \dots & \mu_{2n} \end{vmatrix} = 0 \quad \dots\dots\dots(11),$$

which is really the expression of the n-th approximating parabola

$$-\tilde{y} + P_n = 0 \quad \dots\dots\dots(12)$$

required. It is clear, that the sign of the terms in the first column can be altered.

We can now proceed to the expression of the n-th correlation ratio.

With these words we denote an expression, which is connected with the n-th parabola of regression (12), in the same manner as the coefficient of correlation is connected with the same parabola for $n = 1$ (i.e. with the straight line fitted to the unknown correlation line) and as the well-known correlation ratio η is connected with the actual regression line.

In this way we shall get a measure of the closeness of points (x, y), representing individuals of the population considered, to our n-th parabola of regression.

Such a "natural" measure of closeness is represented by the very familiar integral

$$\Sigma_n^2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} [y - P_n]^2 f(x, y)\, dx dy \quad \dots\dots\dots(13),$$

in which P_n means the polynomial, giving the n-th parabola of regression. If we suppose for a moment that the coefficients of P_n in (13) are variable, the integral (13), which we may now denote by $\Sigma_n'^2$, will be a function of them, and it is easy to see that if we try to dispose of these coefficients to make $\Sigma_n'^2$ a minimum we shall find the same equations (7) to determine them. Therefore Σ_n^2 is the minimum value of $\Sigma_n'^2$, and we shall call it the mean square deviation of individual observations from the n-th regression-parabola, or shortly, the n-th mean square deviation. To get its expression in terms of μ's and λ's, we consider the following function of the a's and a new variate b,

$$\Sigma_{(b)}^2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} [by - P_n]^2 f(x, y)\, dx dy \quad \dots\dots\dots(14).$$

It is clear that $\Sigma_{(b)}^2$ is a homogeneous polynomial of the second degree in the a's and b. Therefore, using the Euler formula,

$$2\Sigma_{(b)}^2 = \frac{\partial \Sigma_{(b)}^2}{\partial b} b + \frac{\partial \Sigma_{(b)}^2}{\partial a_0} a_0 + \frac{\partial \Sigma_{(b)}^2}{\partial a_1} a_1 + \dots + \frac{\partial \Sigma_{(b)}^2}{\partial a_n} a_n \quad \dots\dots\dots(15),$$

for all values of the a's and b. But putting $b = 1$, we have $\Sigma_{(b)}^2 = \Sigma_n^2$, and as the a's satisfy the conditions (7), i.e. as all

$$\frac{\partial \Sigma_n^2}{\partial a_n} = 0 \quad \dots\dots\dots(16),$$

the equation (15) becomes, for $b = 1$,

$$2\Sigma_n^2 = \frac{\partial \Sigma_{(1)}^2}{\partial b} = 2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} y\,(y - P_n) f(x, y)\,dx\,dy \quad \ldots\ldots\ldots\ldots (17).$$

Using in the ordinary notation σ_y for the standard deviation of y, or putting

$$\sigma_y^2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} y^2 f(x, y)\,dx\,dy \quad \ldots\ldots\ldots\ldots (18),$$

and dividing the two parts of (17) by 2, we have

$$\Sigma_n^2 = \sigma_y^2 - a_0\lambda_0 - a_1\lambda_1 - \ldots - a_k\lambda_k - \ldots - a_n\lambda_k \quad \ldots\ldots\ldots\ldots (19).$$

Comparing this expression with (12), we see that to get Σ_n^2 we have only to put into (11) σ_y^2 instead of y and λ_k instead of x^k ($k = 0, 1, 2 \ldots n$), and to divide by the coefficient of $\tilde{y}$, i.e. by $-\Delta_n$. In this way we have

$$\Sigma_n^2 = \frac{1}{\Delta_n}\begin{vmatrix} \sigma_y^2, & \lambda_0, & \lambda_1, & \ldots & \lambda_k, & \ldots & \lambda_n \\ \lambda_0, & \mu_0, & \mu_1, & \ldots & \mu_k, & \ldots & \mu_n \\ \lambda_1, & \mu_1, & \mu_2, & \ldots & \mu_{k+1}, & \ldots & \mu_{n+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \lambda_n, & \mu_n, & \mu_{n+1}, & \ldots & \mu_{n+k}, & \ldots & \mu_{2n} \end{vmatrix} \quad \ldots\ldots\ldots\ldots (20),$$

or

$$\Sigma_n^2 = \sigma_y^2 (1 - H_n^2) \quad \ldots\ldots\ldots\ldots (21),$$

where

$$H_n^2 = \frac{1}{\sigma_y^2 \Delta_n}\begin{vmatrix} 0, & \lambda_0, & \lambda_1, & \ldots & \lambda_k, & \ldots & \lambda_n \\ -\lambda_0, & \mu_0, & \mu_1, & \ldots & \mu_k, & \ldots & \mu_n \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ -\lambda_k, & \mu_k, & \mu_{k+1}, & \ldots & \mu_{2k}, & \ldots & \mu_{k+n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ -\lambda_n, & \mu_n, & \mu_{n+1}, & \ldots & \mu_{n+k}, & \ldots & \mu_{2n} \end{vmatrix} \quad \ldots\ldots\ldots\ldots (22),$$

which is the expression of the n-th correlation ratio we wanted to find. It is easy to see that for $n = 1$, we have

$$H_1^2 = \frac{\lambda_1^2}{\sigma_y^2 \mu_2} \quad \ldots\ldots\ldots\ldots (23),$$

which is the expression of the ordinary coefficient of correlation squared, r_{xy}^2*. We see further, as the equation of the n-th regression parabola (11) can be also written in the following manner,

$$\tilde{y} = \begin{vmatrix} 0, & 1, & x, & x^2, & \ldots & x^k, & \ldots & x^n \\ -\lambda_0, & \mu_0, & \mu_1, & \mu_2, & \ldots & \mu_k, & \ldots & \mu_n \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ -\lambda_k, & \mu_k, & \mu_{k+1}, & \mu_{k+2}, & \ldots & \mu_{2k}, & \ldots & \mu_{k+n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ -\lambda_n, & \mu_n, & \mu_{n+1}, & \mu_{n+2}, & \ldots & \mu_{n+k}, & \ldots & \mu_{2n} \end{vmatrix} \frac{1}{\Delta_n} = \frac{\Delta_n'}{\Delta_n} \text{ (say)} \ldots (24),$$

* As H_n^2 is nothing but the most natural generalization of the coefficient of correlation squared, it would be perhaps better to call it "the n-th coefficient of correlation." Unfortunately this name is adapted for what I should call "the coefficient of correlation between n-th powers of variates."

that, having the equation of the n-th regression-parabola, *we get at once the expression of* H_n^2 *by putting* H_n^2 *instead of* $\tilde{y}$ *and* $\frac{\lambda_k}{\sigma_y^2}$ *instead of the* k*-th power of* x $(k = 0, 1 \ldots n)$. *It is necessary to remember that* x *and* y *are measured from their means.*

Expansions of P_n *and* H_n^2.

3. We shall find it of interest to calculate the general expression for the difference

$$Q_n = P_n - P_{n-1} \quad \ldots\ldots(25).$$

As $P_0 = 0$, we shall be able to express the right-hand side of the equation of the n-th regression-parabola in the form of a sum

$$P_n = Q_1 + Q_2 + \ldots + Q_n \quad \ldots\ldots(26).$$

In this way we shall reach the expressions for successive regression-parabolas, without any appeal to the theory of continued fractions and of orthogonal functions. For $n \leqslant 4$ these expressions have been given by Prof. K. Pearson in *Biometrika*, *loc. cit.*

We shall need here the following property of determinants. Let D be a determinant and $D_{rs\ldots t|ij\ldots k}$ the determinant which we obtain from D by cancelling the r-th, s-th…t-th rows and the same number of columns, namely the i-th, j-th…k-th columns.

We have $$D \times D_{1m|1m} = D_{1|1} D_{m|m} - D_{1|m} D_{m|1} \quad \ldots\ldots(27),$$

where $m > 1$.

If $D_{1|1}$ and $D_{1m|1m}$ are not equal to zero, we can write

$$\frac{D}{D_{1|1}} - \frac{D_{m|m}}{D_{1m|1m}} = - \frac{D_{m|1} D_{1|m}}{D_{1|1} D_{1m|1m}} \quad \ldots\ldots(28),$$

the identity which we shall apply to our expressions of H_n^2 and P_n.

Turning back to (25) and putting Δ' instead of Δ_n', we can now write

$$Q_n = P_n - P_{n-1} = \frac{\Delta'}{\Delta'_{1|1}} - \frac{\Delta'_{n|n}}{\Delta'_{1n|1n}} \quad \ldots\ldots(29),$$

and, using (28),

$$Q_n = - \frac{1}{\Delta_{n-1} \Delta_n} \begin{vmatrix} -\lambda_0, & \mu_0, & \mu_1, & \ldots & \mu_{n-1} \\ -\lambda_1, & \mu_1, & \mu_2, & \ldots & \mu_n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ -\lambda_n, & \mu_n, & \mu_{n+1}, & \ldots & \mu_{2n-1} \end{vmatrix} \times \begin{vmatrix} 1, & x, & x^2, & \ldots & x^n \\ \mu_0, & \mu_1, & \mu_2, & \ldots & \mu_n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \mu_{n-1}, & \mu_n, & \mu_{n+1}, & \ldots & \mu_{2n-1} \end{vmatrix} \quad \ldots\ldots(30),$$

or changing the columns of the first determinant into rows and changing the sign of the λ's:

$$Q_n = \frac{1}{\Delta_{n-1} \Delta_n} \begin{vmatrix} \lambda_0, & \lambda_1, & \lambda_2, & \ldots & \lambda_n \\ \mu_0, & \mu_1, & \mu_2, & \ldots & \mu_n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \mu_{n-1}, & \mu_n, & \mu_{n+1}, & \ldots & \mu_{2n-1} \end{vmatrix} \times \begin{vmatrix} 1, & x, & x^2, & \ldots & x^n \\ \mu_0, & \mu_1, & \mu_2, & \ldots & \mu_n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \mu_{n-1}, & \mu_n, & \mu_{n+1}, & \ldots & \mu_{2n-1} \end{vmatrix} \quad \ldots\ldots(31).$$

We see that the second determinant can be obtained from the first one by putting x^k for λ^k $(k = 0, 1 \dots n)$. The first determinant we shall denote by V_n, and the second by $V_n(x)$. Now the equation of the n-th regression-parabola can be written as follows:

$$\bar{y} = \frac{V_1 V_1(x)}{\Delta_0 \Delta_1} + \frac{V_2 V_2(x)}{\Delta_1 \Delta_2} + \dots + \frac{V_n V_n(x)}{\Delta_{n-1} \Delta_n} \dots\dots\dots\dots\dots\dots (32).$$

Applying the same method to (22), we easily find the expansion of H_n^2, namely,

$$H_n^2 = \frac{1}{\sigma_y^2}\left[\frac{V_1^2}{\Delta_0 \Delta_1} + \frac{V_2^2}{\Delta_1 \Delta_2} + \dots + \frac{V_n^2}{\Delta_{n-1} \Delta_n}\right] \dots\dots\dots\dots\dots\dots (33).$$

As the calculation of higher parabolas and correlation ratios needs the knowledge of the higher moments of the sampled population, the above results will be of interest in theoretical research, where the higher moments may be supposed to be known. Such a case arises, for instance, when we consider the dependency of the standard deviation upon the mean of a sample.

The expansion of P_n can be used to find the necessary and sufficient conditions which the constants λ and μ must satisfy in order that the actual regression line may be represented by an n-th order parabola.

Assume that $\bar{y} = P_n$ .....(34)

represents the actual regression line. Then, using (24), we find

$$\lambda_k = \int_{-\infty}^{+\infty} \bar{y}_x x^k p(x)\, dx = -\frac{1}{\Delta_n} \begin{vmatrix} 0, & \mu_k, & \mu_{k+1}, \dots & \mu_{n+k} \\ \lambda_0, & \mu_0, & \mu_1, \dots & \mu_n \\ \lambda_1, & \mu_1, & \mu_2, \dots & \mu_{n+1} \\ \dots & \dots & \dots & \dots \\ \lambda_n, & \mu_n, & \mu_{n+1}, \dots & \mu_{2n} \end{vmatrix} \quad \dots\dots\dots (35),$$

where the coefficient of μ_{n+k} is not zero, and this is the necessary condition required. Assuming that the expansion (32) is permissible, we shall now prove that this condition is sufficient. In fact, assume that all the λ_k's are linear functions of $\mu_k, \mu_{k+1}, \dots \mu_{n+k}$ represented by (35). Considering V_{n+s}, where $s \geqslant 1$, we see that the terms of the first line of this determinant can be represented by the *same* linear and homogeneous function (35) of corresponding terms of the $n+1$ following lines of the same determinant. We conclude that $V_{n+s} = 0$. Now if we consider V_n and if we multiply the terms of its t-th row by the coefficient of μ_{k+t-2} $(t = 2, 3 \dots n+1)$ in (35) and subtract the products from the terms of the first line, we shall see that owing to (35) the determinant V_n will be transformed into the product of Δ_n into the coefficient of μ_{n+k} in (35), both of them being different from zero. Therefore $V_n \neq 0$. As the coefficient of x^n in $V_n(x)$ is $\Delta_{n-1} \neq 0$, we see that if all the λ's satisfy the condition (35), where the coefficient of μ_{n+k} is not zero, the actual regression line is represented by an n-th order parabola.

The general condition established can be considered as the generalisation of respective formulae given by K. Pearson for $n \leqslant 4$ (*loc. cit.*), and by A. A. Tchouproff* for $n \leqslant 2$.

* *Grundbegriffe u. Grundprobleme der Korrelationstheorie*, Teubner, 1925, pp. 48—49.

ON THE PROBABLE ERROR OF THE MODE OF SKEW FREQUENCY DISTRIBUTIONS.

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(Thesis approved for the Degree of Doctor of Science in the University of London.)

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Introduction.

This paper consists of four Chapters. The first treats of the deduction of a formula giving the probable error of the mode of Pearsonian curves as a function of β_1 and β_2, the second deals with the requisite tables, the third represents the probable error of the mode as a surface, while the fourth Chapter is concerned with statistical applications. In spite of the importance of the mode no general investigation of the accuracy with which it can be determined has hitherto been made, although Professors Pearson and Filon* have indicated by a very different method its determination for special curves. I have endeavoured in this paper to find a formula for the probable error of the mode, to make a table of it for an adequate range, to draw a contour diagram and to make a model of the surface in order to visualize the whole field. The contour diagram provides facilities for finding an approximate value; a more accurate value can be obtained by interpolation from the table itself.

* *Phil. Trans.* Vol. 191 (1898), pp. 229—311.

Chapter I. Deduction of a Formula to calculate the Probable Error of the Mode.

Let the differential equation of the frequency curve be assumed to be given by the form:

$$\frac{1}{y}\frac{dy}{dx} = -\frac{x+a}{c_0 + c_1 x + c_2 x^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Multiplying (1) by $\quad yx^s(c_0 + c_1 x + c_2 x^2)\,dx$

and integrating throughout the range b_1 to b_2 we get

$$\int_{b_1}^{b_2} x^s (c_0 + c_1 x + c_2 x^2)\,dy = -\int_{b_1}^{b_2} y\,(x^{s+1} + a x^s)\,dx \quad \ldots\ldots\ldots\ldots (2).$$

Integrating the left-hand side of (2) by parts and remembering that usually the expression $yx^s(c_0 + c_1 x + c_2 x^2)$ vanishes at both ends of the range we get:

$$-sc_0\int_{b_1}^{b_2} x^{s-1}\,y\,dx - (s+1)\,c_1\int_{b_1}^{b_2} x^s y\,dx - (s+2)\,c_2\int_{b_1}^{b_2} x^{s+1}\,y\,dx$$
$$= -\int_{b_1}^{b_2} x^{s+1}\,y\,dx - a\int_{b_1}^{b_2} x^s y\,dx \quad \ldots\ldots\ldots\ldots (3),$$

or with the usual notation for the moment coefficients

$$sc_0\mu'_{s-1} + (s+1)\,c_1\mu_s' + (s+2)\,c_2\,\mu'_{s+1} = \mu'_{s+1} + a\mu_s' \quad \ldots\ldots\ldots\ldots (4).$$

Hence, the origin being any fixed point, we have for:

$$s = 0, \qquad c_1\mu_0' + 2c_2\mu_1' = \mu_1' + a\mu_0',$$

or,
$$c_1 + 2c_2\mu_1' = \mu_1' + a \quad \ldots\ldots\ldots\ldots (5),$$

$$s = 1, \qquad c_0 + 2c_1\mu_1' + 3c_2\mu_2' = \mu_2' + a\mu_1' \quad \ldots\ldots\ldots\ldots (6),$$

$$s = 2, \qquad 2c_0\mu_1' + 3c_1\mu_2' + 4c_2\mu_3' = \mu_3' + a\mu_2' \quad \ldots\ldots\ldots\ldots (7),$$

$$s = 3, \qquad 3c_0\mu_2' + 4c_1\mu_3' + 5c_2\mu_4' = \mu_4' + a\mu_3' \quad \ldots\ldots\ldots\ldots (8).$$

Eliminating c_0, c_1 and c_2 from (5), (6), (7) and (8):

$$a = \frac{\mu_4'(\mu_3' - 13\mu_2'\mu_1' + 12\mu_1'^3) + \mu_3'(8\mu_2'\mu_1' - 20\mu_2'\mu_1'^2 + 3\mu_2'^2) + 9\mu_2'^3\mu_1'}{10\mu_4'\mu_2' - 10\mu_4'\mu_1'^2 - 12\mu_3'^2 + 32\mu_2'\mu_2'\mu_1' - 8\mu_2'\mu_1'^3 - 18\mu_2'^3 + 6\mu_2'^2\mu_1'^2} \equiv \frac{N}{D}.$$

Denote the distance of the mode from any fixed origin by M:

$$M \equiv \frac{N}{D} = a \quad \ldots\ldots\ldots\ldots (9),$$

$$\log M = \log N - \log D \quad \ldots\ldots\ldots\ldots (10).$$

Now assuming the sample so large that statistical differentials may be replaced by mathematical differentials we reach by differentiating (10):

$$\frac{\delta M}{M} = \delta\mu_4'\left\{\frac{\bar\mu_3' - 13\bar\mu_2'\bar\mu_1' + 12\bar\mu_1'^3}{\bar N} - \frac{10\,(\bar\mu_2' - \bar\mu_1'^2)}{\bar D}\right\}$$
$$+ \delta\mu_3'\left\{\frac{\bar\mu_4' + 16\bar\mu_3'\bar\mu_1' - 20\bar\mu_2'\bar\mu_1'^2 + 3\bar\mu_2'^2}{\bar N} + \frac{24\bar\mu_3' - 32\bar\mu_2'\bar\mu_1' + 8\bar\mu_1'^3}{\bar D}\right\}$$

$$-\delta\mu_2'\left\{\frac{13\bar{\mu}_4'\bar{\mu}_1'+20\bar{\mu}_3'\bar{\mu}_1'^2-6\bar{\mu}_2'\bar{\mu}_3'-27\bar{\mu}_2'^2\bar{\mu}_1'}{\bar{N}}\right.$$
$$\left.+\frac{10\bar{\mu}_4'+32\bar{\mu}_3'\bar{\mu}_1'-54\bar{\mu}_2'^2+12\bar{\mu}_2'\bar{\mu}_1'^2}{\bar{D}}\right\}$$
$$-\delta\mu_1'\left\{\frac{13\bar{\mu}_4'\bar{\mu}_2'-36\bar{\mu}_4'\bar{\mu}_1'^2-8\bar{\mu}_3'^2+40\bar{\mu}_3'\bar{\mu}_2'\bar{\mu}_1'-9\bar{\mu}_2'^2}{\bar{N}}\right.$$
$$\left.+\frac{-20\bar{\mu}_4'\bar{\mu}_1'+32\bar{\mu}_3'\bar{\mu}_2'-24\bar{\mu}_3'\bar{\mu}_1'^2+12\bar{\mu}_2'^2\bar{\mu}_1'}{\bar{D}}\right\} \quad \ldots\ldots(11),$$

where all the $\bar{\mu}$'s are for the sampled population; we shall indicate by rules over our symbols that they are the values of the corresponding quantities for this population. If now we refer to the mean of the sampled population as our fixed origin we shall have $\bar{\mu}_1' = 0$ and

$$\bar{N} = \bar{\mu}_4\bar{\mu}_3 + 3\bar{\mu}_2^2\bar{\mu}_3 = \bar{\sigma}^7\sqrt{\bar{\beta}_1}(\bar{\beta}_2+3) \quad \ldots\ldots(12),$$
$$\bar{D} = 10\bar{\mu}_4\bar{\mu}_2 - 12\bar{\mu}_3^2 - 18\bar{\mu}_2^3 = 2\bar{\sigma}^8(5\bar{\beta}_2 - 6\bar{\beta}_1 - 9) \quad \ldots\ldots(13),$$
$$\bar{M} \equiv \frac{\bar{N}}{\bar{D}} = \frac{\bar{\sigma}}{2}\frac{\sqrt{\bar{\beta}_1}(\bar{\beta}_2+3)}{(5\bar{\beta}_2 - 6\bar{\beta}_1 - 9)} \quad \ldots\ldots(14).$$

Introducing (12), (13) and (14) into (11):

$$\frac{\delta M}{\bar{M}} = \frac{\delta\mu_4'}{\bar{\sigma}^4}\left\{\frac{1}{\bar{\beta}_2+3} - \frac{5}{5\bar{\beta}_2 - 6\bar{\beta}_1 - 9}\right\} + \frac{\delta\mu_3'}{\bar{\sigma}^3}\left\{\frac{1}{\sqrt{\bar{\beta}_1}} + \frac{12\sqrt{\bar{\beta}_1}}{5\bar{\beta}_2 - 6\bar{\beta}_1 - 9}\right\}$$
$$\frac{\delta\mu_2'}{\bar{\sigma}^2}\left\{\frac{6}{\bar{\beta}_2+3} - \frac{5\bar{\beta}_2 - 27}{5\bar{\beta}_2 - 6\bar{\beta}_1 - 9}\right\} + \frac{\delta\mu_1'}{\bar{\sigma}}\left\{\frac{8\bar{\beta}_1 - 13\bar{\beta}_2 + 9}{\sqrt{\bar{\beta}_1}(\bar{\beta}_2+3)} - \frac{16\sqrt{\bar{\beta}_1}}{5\bar{\beta}_2 - 6\bar{\beta}_1 - 9}\right\} \ldots(15),$$

$$\delta M = \tfrac{1}{2}\delta\mu_1'\left\{\frac{8\bar{\beta}_1 - 13\bar{\beta}_2 + 9}{5\bar{\beta}_2 - 6\bar{\beta}_1 - 9} - \frac{16\bar{\beta}_1(\bar{\beta}_2+3)}{(5\bar{\beta}_2 - 6\bar{\beta}_1 - 9)^2}\right\}$$
$$+\tfrac{1}{2}\frac{\delta\mu_2'}{\bar{\sigma}}\left\{\frac{6\sqrt{\bar{\beta}_1}}{5\bar{\beta}_2 - 6\bar{\beta}_1 - 9} - \frac{\sqrt{\bar{\beta}_1}(\bar{\beta}_2+3)(5\bar{\beta}_2 - 27)}{(5\bar{\beta}_2 - 6\bar{\beta}_1 - 9)^2}\right\}$$
$$+\tfrac{1}{2}\frac{\delta\mu_2'}{\bar{\sigma}^2}\left\{\frac{\bar{\beta}_2+3}{5\bar{\beta}_2 - 6\bar{\beta}_1 - 9} + \frac{12\bar{\beta}_1(\bar{\beta}_2+3)}{(5\bar{\beta}_2 - 6\bar{\beta}_1 - 9)^2}\right\}$$
$$+\tfrac{1}{2}\frac{\delta\mu_4'}{\bar{\sigma}^3}\left\{\frac{\sqrt{\bar{\beta}_1}}{5\bar{\beta}_2 - 6\bar{\beta}_1 - 9} - \frac{5\sqrt{\bar{\beta}_1}(\bar{\beta}_2+3)}{(5\bar{\beta}_2 - 6\bar{\beta}_1 - 9)^2}\right\} \quad \ldots\ldots(16).$$

Now if n be the number of individuals in the sample and we represent by curled brackets the mean value for all possible samples of the expression enclosed, then:

$$\{\delta\mu_1'^2\} = \frac{\bar{\sigma}^2}{n} \quad \ldots\ldots(17), \qquad \{\delta\mu_2'^2\} = (\bar{\beta}_2 - 1)\frac{\bar{\sigma}^4}{n} \quad \ldots\ldots(18),$$

$$\{\delta\mu_3'^2\} = \frac{\bar{\mu}_6 - \bar{\mu}_3^2}{n} = \frac{(\bar{\beta}_4 - \bar{\beta}_1)\bar{\sigma}^6}{n} \quad \ldots\ldots(19), \qquad \{\delta\mu_4'^2\} = \frac{\bar{\mu}_8 - \bar{\mu}_4^2}{n} = \frac{(\bar{\beta}_6 - \bar{\beta}_2^2)\bar{\sigma}^8}{n} \quad \ldots\ldots(20),$$

$$\{\delta\mu_1'\delta\mu_2'\} = \frac{\bar{\mu}_3 - \bar{\mu}_2\bar{\mu}_1}{n} = \frac{\sqrt{\bar{\beta}_1}\bar{\sigma}^3}{n} \quad \ldots\ldots(21), \qquad \{\delta\mu_1'\delta\mu_3'\} = \frac{\bar{\mu}_4 - \bar{\mu}_3\bar{\mu}_1}{n} = \frac{\bar{\beta}_2\bar{\sigma}^4}{n} \quad \ldots\ldots(22),$$

$$\{\delta\mu_1'\delta\mu_4'\} = \frac{\overline{\mu}_5 - \overline{\mu}_4\overline{\mu}_1}{n} = \frac{\overline{\beta}_3}{\sqrt{\overline{\beta}_1}}\frac{\overline{\sigma}^5}{n} \quad \ldots\ldots(23), \qquad \{\delta\mu_2'\delta\mu_3'\} = \frac{\overline{\mu}_5 - \overline{\mu}_2\overline{\mu}_3}{n} = \frac{\overline{\beta}_3 - \overline{\beta}_1}{\sqrt{\overline{\beta}_1}}\frac{\overline{\sigma}^5}{n} \quad \ldots(24),$$

$$\{\delta\mu_3'\delta\mu_4'\} = \frac{\overline{\mu}_7 - \overline{\mu}_3\overline{\mu}_4}{n} = \frac{\overline{\beta}_5 - \overline{\beta}_1\overline{\beta}_2}{\sqrt{\overline{\beta}_1}}\frac{\overline{\sigma}^7}{n} \ldots(25), \qquad \{\delta\mu_2'\delta\mu_4'\} = \frac{\overline{\mu}_6 - \overline{\mu}_2\overline{\mu}_4}{n} = \frac{\overline{\beta}_4 - \overline{\beta}_2}{n}\overline{\sigma}^6 \quad \ldots(26),$$

since $\mu_1 = 0$ for the sampled population.

Further let

$$\left.\begin{aligned} B_1 &= \tfrac{1}{2}\left\{\frac{8\overline{\beta}_1 - 13\overline{\beta}_2 + 9}{5\overline{\beta}_2 - 6\overline{\beta}_1 - 9} - \frac{16\overline{\beta}_1(\overline{\beta}_2 + 3)}{(5\overline{\beta}_2 - 6\overline{\beta}_1 - 9)^2}\right\} \\ B_2 &= \tfrac{1}{2}\left\{\frac{6}{5\overline{\beta}_2 - 6\overline{\beta}_1 - 9} - \frac{(\overline{\beta}_2 + 3)(5\overline{\beta}_2 - 27)}{(5\overline{\beta}_2 - 6\overline{\beta}_1 - 9)^2}\right\} \\ B_3 &= \tfrac{1}{2}\left\{\frac{\overline{\beta}_2 + 3}{5\overline{\beta}_2 - 6\overline{\beta}_1 - 9} + \frac{12\overline{\beta}_1(\overline{\beta}_2 + 3)}{(5\overline{\beta}_2 - 6\overline{\beta}_1 - 9)^2}\right\} \\ B_4 &= \tfrac{1}{2}\left\{\frac{1}{5\overline{\beta}_2 - 6\overline{\beta}_1 - 9} - \frac{5(\overline{\beta}_2 + 3)}{(5\overline{\beta}_2 - 6\overline{\beta}_1 - 9)^2}\right\} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(27).$$

Then we may write:

$$\delta M = \delta\mu_1' B_1 + \frac{\delta\mu_2'}{\overline{\sigma}}\sqrt{\overline{\beta}_1}B_2 + \frac{\delta\mu_3'}{\overline{\sigma}^2}B_3 + \frac{\delta\mu_4'}{\overline{\sigma}^3}\sqrt{\overline{\beta}_1}B_4.$$

Square, sum for all possible samples, divide by their number, and we have:

$$\begin{aligned} \sigma_M^2 = \frac{\overline{\sigma}^2}{n}\{&B_1^2 + \overline{\beta}_1(\overline{\beta}_2 - 1)B_2^2 + (\overline{\beta}_4 - \overline{\beta}_1)B_3^2 + \overline{\beta}_1(\overline{\beta}_6 - \overline{\beta}_2^2)B_4^2 + 2B_1B_2\overline{\beta}_1 \\ &+ 2B_1B_3\overline{\beta}_2 + 2B_1B_4\overline{\beta}_3 + 2B_2B_3(\overline{\beta}_3 - \overline{\beta}_1) + 2B_2B_4\overline{\beta}_1(\overline{\beta}_4 - \overline{\beta}_2) \\ &+ 2B_3B_4(\overline{\beta}_5 - \overline{\beta}_1\overline{\beta}_2)\} \quad \ldots\ldots\ldots(28). \end{aligned}$$

Hence we deduce that:

Ratio of the probable error of the mode to the probable error of the mean

$$\begin{aligned} = \{&B_1^2 + \overline{\beta}_1(\overline{\beta}_2 - 1)B_2^2 + (\overline{\beta}_4 - \overline{\beta}_1)B_3^2 + \overline{\beta}_1(\overline{\beta}_6 - \overline{\beta}_2^2)B_4^2 + 2B_1B_2\overline{\beta}_1 \\ &+ 2B_3B_4(\overline{\beta}_5 - \overline{\beta}_1\overline{\beta}_2) + 2B_1B_3\overline{\beta}_2 + 2B_1B_4\overline{\beta}_3 + 2B_2B_3(\overline{\beta}_3 - \overline{\beta}_1) \\ &+ 2B_2B_4(\overline{\beta}_4 - \overline{\beta}_2)\overline{\beta}_1\}^{\frac{1}{2}} \quad \ldots\ldots(29). \end{aligned}$$

Chapter II. Tabling of the Probable Error of the Mode.

(1) *Explanation of Tables.* It will be clear from this result that the determination of the probable error (or of the standard deviation) of the mode for each individual case would involve a large amount of computation. In order to avoid this it is necessary that the purely algebraical formula (29) should be replaced by tables. An examination of the values of B_1, B_2, B_3 and B_4 shows that they involve only β_1 and β_2, but the presence of the factor $1/(5\overline{\beta}_2 - 6\overline{\beta}_1 - 9)$ indicates that the probable error of the mode will become excessive in the neighbourhood of the line $5\overline{\beta}_2 - 6\overline{\beta}_1 - 9 = 0$. But this is the "Rectangular Line" passing through the "Rectangular Point" $\beta_1 = 0$, $\beta_2 = 1{\cdot}8$, and is the so-called "axis" of the biquadratic which encloses J-curves and separates limited range curves from U-curves. When we get curves falling into the J-area our formula must break down, for the curve

starts either with a finite or an infinite ordinate, which in a different sense is then the "mode." The only variation in the position of this maximum ordinate is that due to change in the terminal of the range from sample to sample; but this is not given by our formula, which now gives a spurious mode lying outside the curve range. On the border of the biquadratic curve which encloses true J-curves the spurious mode becomes real. Along the "rectangular line" itself the curve suddenly changes from a right-handed J to a left-handed ζ, and under such circumstances the spurious mode, as we approach the rectangular line, swings across and its probable error becomes infinite. This is the meaning of the factor $1/(5\bar{\beta}_2 - 6\bar{\beta}_1 - 9)$ in the B-terms, but the interest of this spurious mode is purely mathematical and not statistical. The biquadrate bounding the area of the spurious mode is not shown on Diagram I for it is too crowded; it passes near to contour 15 above and enclosing the asymptote reaches to about contour 7 below it.

Passing now to a further point we notice that $\bar{\beta}_3$, $\bar{\beta}_4$, $\bar{\beta}_5$ and $\bar{\beta}_6$ occur in the values of the mean powers and mean products of the moment coefficients. It was, therefore, needful to compute these, or to use the values given in *Biometrika*, Vol. VII. by A. J. Rhind for these quantities, of course assuming always that a Pearson frequency curve might be used. But on examination it was found (*a*) that there were certain errors in Rhind's Tables, and (*b*) what is more important, that smaller intervals of the argument β_2 were needful. Accordingly new Tables of the values of β_3, β_4, β_5 and β_6 for given values of β_1 and β_2 were computed from the difference formulae:

$$\beta_{2s} = (2s+1)\{\tfrac{1}{2}\beta_{2s-1} + (1+\tfrac{1}{2}\alpha)\beta_{2s-2}\}/\{1-\tfrac{1}{2}(2s-1)\alpha\},$$

$$\beta_{2s+1} = (2s+2)\{\tfrac{1}{2}\beta_1\beta_{2s} + (1+\tfrac{1}{2}\alpha)\beta_{2s-1}\}/\{1-\tfrac{1}{2}(2s)\alpha\},$$

where
$$\alpha = (2\beta_2 - 3\beta_1 - 6)/(\beta_2 + 3) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(30).$$

(2) The calculation of this new Table I, which the author believes will be of service for other purposes, occupied several months of strenuous labour.

The next stage in the work was the computing of B_1, B_2, B_3 and B_4; their values are not reproduced here, because it is not clear that they will be of service for other investigations. The values of the standard deviations and mean products are so easily obtained from Equations (17)—(26) that it does not seem worth while providing tables of them, although they occur frequently in practical work.

(3) By aid of the Tables of the β's and the B's Table II giving the ratio of the probable error of the mode to that of the mean was constructed. It will be noted that in the final formula (29) there is no occurrence of $\sqrt{\bar{\beta}_1}$, which is found in (21), (23), (24) and (25), but drops out on the squaring of δM and the substitution of the products of the moment coefficients.

I suggest that the Table now provided will enable future workers to give the standard deviation of the mode with the same ease as in the past the standard deviation of the mean has been given, and thus to complete the specification of the constants of skew frequency distributions for *large* samples. I recognise that my work does not enable one to determine the frequency distribution of the mode, but

TABLE I*.

Values of β_2.

β_2.	β_1. 0	·1	·2	·3	·4	·5	·6	·7
1·8	0	·41440	·78154	1·10667	1·39429	1·64888	1·87200	2·06839
1·9	0	·44892	·84727	1·20075	1·51422	1·79186	2·03730	2·25370
2·0	0	·48493	·91579	1·29873	1·63902	1·94118	2·20909	2·44615
2·1	0	·52250	·98720	1·40077	1·76889	2·09643	2·38759	2·64600
2·2	0	·56169	1·06162	1·50701	1·90400	2·25783	2·57302	2·85348
2·3	0	·60257	1·13918	1·61763	2·04456	2·42561	2·76565	3·06886
2·4	0	·64522	1·22000	1·73280	2·19077	2·60000	2·96571	3·29241
2·5	0	·68971	1·30423	1·85270	2·34286	2·78125	3·17349	3·52442
2·6	0	·73612	1·39200	1·97753	2·50105	2·96962	3·38927	3·76518
2·7	0	·78455	1·48348	2·10750	2·66560	3·16538	3·61333	4·01500
2·8	0	·83508	1·57882	2·24282	2·83676	3·36883	3·84600	4·27422
2·9	0	·88781	1·67821	2·38371	3·01479	3·58026	4·08759	4·54317
3·0	0	·94286	1·78182	2·53043	3·20000	3·80000	4·33846	4·82222
3·1	0	1·00032	1·88985	2·68324	3·39268	4·02838	4·59896	5·11175
3·2	0	1·06033	2·00250	2·84239	3·59314	4·26575	4·86947	5·41215
3·3	0	1·12300	2·12000	3·00818	3·80174	4·51250	5·15040	5·72385
3·4	0	1·18847	2·24258	3·18092	4·01882	4·76901	5·44216	6·04727
3·5	0	1·25690	2·37049	3·36094	4·24478	5·03571	5·74521	6·38289
3·6	0	1·32842	2·50400	3·54857	4·48000	5·31304	6·06000	6·73120
3·7	0	1·40321	2·64339	3·74419	4·72492	5·60147	6·38704	7·09270
3·8	0	1·48145	2·78897	3·94820	4·98000	5·90149	6·72686	7·46795
3·9	0	1·56333	2·94105	4·16100	5·24571	6·21364	7·08000	7·85750
4·0	0	1·64906	3·10000	4·38305	5·52258	6·53846	7·44706	8·26197
4·1	0	1·73885	3·26618	4·61483	5·81115	6·87656	7·82866	8·68200
4·2	0	1·83294	3·44000	4·85684	6·11200	7·22857	8·22545	9·11826
4·3	0	1·93160	3·62189	5·10964	6·42576	7·59516	8·63815	9·57147
4·4	0	2·03510	3·81231	5·37382	6·75310	7·97705	9·06750	10·04239
4·5	0	2·14375	4·01176	5·65000	7·09474	8·37500	9·51429	10·53182
4·6	—	2·25787	4·22080	5·93887	7·45143	8·78983	9·97935	11·04062
4·7	—	—	4·44000	6·24115	7·82400	9·22241	10·46360	11·56969
4·8	—	—	4·67000	6·55765	8·21333	9·67368	10·96800	12·12000
4·9	—	—	—	6·88920	8·62038	10·14464	11·49356	12·69258
5·0	—	—	—	7·23673	9·04615	10·63636	12·04138	13·28852
5·1	—	—	—	—	9·49176	11·15000	12·61263	13·90900
5·2	—	—	—	—	9·95840	11·68679	13·20857	14·55525
5·3	—	—	—	—	—	12·24808	13·83055	15·22862
5·4	—	—	—	—	—	12·83529	14·48000	15·93053
5·5	—	—	—	—	—	—	15·15849	16·66250
5·6	—	—	—	—	—	—	15·86769	17·42618
5·7	—	—	—	—	—	—	—	18·22333
5·8	—	—	—	—	—	—	—	19·05585
5·9	—	—	—	—	—	—	—	—
6·0	—	—	—	—	—	—	—	—
6·1	—	—	—	—	—	—	—	—
6·2	—	—	—	—	—	—	—	—
6·3	—	—	—	—	—	—	—	—
6·4	—	—	—	—	—	—	—	—
6·5	—	—	—	—	—	—	—	—
6·6	—	—	—	—	—	—	—	—
6·7	—	—	—	—	—	—	—	—
6·8	—	—	—	—	—	—	—	—
6·9	—	—	—	—	—	—	—	—
7·0	—	—	—	—	—	—	—	—

* In the values given for β_5 and β_6 complete reliability beyond the eighth figure is not to be depended on.

TABLE I—(*continued*).

Values of β_3.

β_1.

·8	·9	1·0	1·1	1·2	1·3	1·4	1·5	
2·24000	2·38909	2·51765	2·62743	2·72000	2·79676	2·85895	2·90769	1·8
2·44379	2·61000	2·75446	2·87904	2·98542	3·07509	3·14938	3·20948	1·9
2·65532	2·83918	3·00000	3·13981	3·26038	3·36330	3·45000	3·52174	2·0
2·87484	3·07688	3·25455	3·41000	3·54514	3·66167	3·76108	3·84474	2·1
3·10261	3·32337	3·51837	3·68990	3·84000	3·97047	4·08291	4·17876	2·2
3·33890	3·57894	3·79175	3·97980	4·14524	4·29000	4·41578	4·52411	2·3
3·58400	3·84387	4·07500	4·28000	4·46118	4·62057	4·76000	4·88108	2·4
3·83820	4·11848	4·36842	4·59082	4·78812	4·96250	5·11589	5·25000	2·5
4·10182	4·40308	4·67234	4·91258	5·12640	5·31612	5·48377	5·63119	2·6
4·37517	4·69800	4·98710	5·24562	5·47636	5·68176	5·86400	6·02500	2·7
4·65860	5·00360	5·31304	5·59032	5·83837	6·05980	6·25692	6·43178	2·8
4·95247	5·32023	5·65055	5·94702	6·21278	6·45060	6·66291	6·85189	2·9
5·25714	5·64828	6·00000	6·31613	6·60000	6·85455	7·08235	7·28571	3·0
5·57301	5·98814	6·36180	6·69804	7·00042	7·27204	7·51564	7·73365	3·1
5·90049	6·34024	6·73636	7·09319	7·41447	7·70351	7·96320	8·19612	3·2
6·24000	6·70500	7·12414	7·50200	7·84258	8·14938	8·42545	8·67353	3·3
6·59200	7·08289	7·52558	7·92494	8·28522	8·61011	8·90286	9·16634	3·4
6·95696	7·47439	7·94118	8·36250	8·74286	9·08617	9·39588	9·67500	3·5
7·33538	7·88000	8·37143	8·81517	9·21600	9·57806	9·90500	10·20000	3·6
7·72779	8·30025	8·81687	9·28349	9·70517	10·08630	10·43074	10·74184	3·7
8·13474	8·73570	9·27805	9·76800	10·21091	10·61143	10·97362	11·30103	3·8
8·55680	9·18692	9·75556	10·26929	10·73379	11·15400	11·53419	11·87812	3·9
8·99459	9·65455	10·25000	10·78795	11·27442	11·71461	12·11304	12·47368	4·0
9·44877	10·13921	10·76203	11·32463	11·83341	12·29386	12·71077	13·08830	4·1
9·92000	10·64160	11·29231	11·88000	12·41143	12·89241	13·32800	13·72258	4·2
10·40901	11·16243	11·84156	12·45475	13·00916	13·51093	13·96539	14·37717	4·3
10·91657	11·70247	12·41053	13·04962	13·62732	14·15012	14·62364	15·05275	4·4
11·44348	12·26250	13·00000	13·66538	14·26667	14·81071	15·30345	15·75000	4·5
11·99059	12·84338	13·61081	14·30286	14·92800	15·49349	16·00558	16·46966	4·6
12·55881	13·44600	14·24384	14·96289	15·61215	16·19927	16·73082	17·21250	4·7
13·14909	14·07130	14·90000	15·64640	16·32000	16·92889	17·48000	17·97931	4·8
13·76246	14·72029	15·58028	16·35432	17·05247	17·68325	18·25398	18·77093	4·9
14·40000	15·39403	16·28571	17·08767	17·81053	18·46329	19·05366	19·58824	5·0
15·06286	16·09364	17·01739	17·84750	18·59520	19·27000	19·88000	20·43214	5·1
15·75226	16·82031	17·77647	18·63493	19·40757	20·10442	20·73400	21·30361	5·2
16·46951	17·57531	18·56418	19·45143	20·24877	20·96763	21·61671	22·20366	5·3
17·21600	18·36000	19·38182	20·29739	21·12000	21·86080	22·52923	23·13333	5·4
17·99322	19·17581	20·23077	21·17500	22·02254	22·78513	23·47273	24·09375	5·5
18·80276	20·02426	21·11250	22·08537	22·95771	23·74192	24·44842	25·08608	5·6
19·64632	20·90700	22·02857	23·03000	23·92696	24·73250	25·45760	26·11154	5·7
20·52571	21·82576	22·98065	24·01046	24·93176	25·75831	26·50162	27·17143	5·8
21·44291	22·78241	23·97049	25·02844	25·97373	26·82086	27·58192	28·26711	5·9
—	23·77895	25·00000	26·08571	27·05455	27·92174	28·70000	29·40000	6·0
—	24·81750	26·07119	27·18419	28·17600	29·06265	29·85746	30·57162	6·1
—	—	27·18621	28·32590	29·34000	30·24537	31·05600	31·78356	6·2
—	—	28·34737	29·51300	30·54857	31·47182	32·29739	33·03750	6·3
—	—	—	30·74780	31·80387	32·74400	33·58353	34·33521	6·4
—	—	—	32·03276	33·10820	34·06406	34·91642	35·67857	6·5
—	—	—	—	34·46400	35·43429	36·29818	37·06957	6·6
—	—	—	—	35·87390	36·85710	37·73108	38·51029	6·7
—	—	—	—	—	38·33508	39·21750	40·00299	6·8
—	—	—	—	—	39·87100	40·76000	41·55000	6·9
—	—	—	—	—	—	42·36129	43·15385	7·0

β_2.

TABLE I—(*continued*).

Values of β_1.

β_2.	β_1. 0	·1	·2	·3	·4	·5	·6	·7
1·8	3·85714	4·07037	4·20199	4·26974	4·28739	4·26566	4·21297	4·13599
1·9	4·40244	4·64442	4·79441	4·87269	4·89492	4·87323	4·81715	4·73416
2·0	5·00000	5·27357	5·44352	5·53307	5·55998	5·53802	5·47791	5·38807
2·1	5·65385	5·96205	6·15366	6·25517	6·28681	6·26413	6·19921	6·10151
2·2	6·36842	6·71453	6·92954	7·04376	7·08000	7·05602	6·98537	6·87863
2·3	7·14865	7·53619	7·77640	7·90384	7·94461	7·91858	7·84109	7·72393
2·4	8·00000	8·43274	8·70000	8·84126	8·88615	8·85714	8·77143	8·64228
2·5	8·92857	9·41053	9·70672	9·86955	9·91071	9·87753	9·81065	9·63899
2·6	9·94118	10·47664	10·80364	10·97368	11·02497	10·98615	10·87892	10·71985
2·7	11·04545	11·63897	11·99861	12·18325	12·23629	12·19003	12·06882	11·89115
2·8	12·25000	12·90637	13·30040	13·49946	13·55280	13·49688	13·35901	13·15981
2·9	13·56452	14·28876	14·71881	14·93176	14·98353	14·91525	14·66914	14·53334
3·0	15·00000	15·79734	16·26482	16·49068	16·53846	16·45455	16·27321	16·02004
3·1	16·56897	17·44479	17·95077	18·18802	18·22872	18·12523	17·91578	17·62896
3·2	18·28571	19·24547	19·79058	20·03698	20·06672	19·93875	19·69601	19·37010
3·3	20·16667	21·21577	21·80000	22·05242	22·06630	21·90809	21·62578	21·25445
3·4	22·23077	23·37444	23·99693	24·25111	24·34303	24·04755	23·71830	23·29418
3·5	24·50000	25·74304	26·40178	26·65201	26·61435	26·37315	25·98826	25·50270
3·6	27·00002	28·34641	29·03790	29·27665	29·20000	28·90287	28·45200	27·89494
3·7	29·76087	31·21363	31·93214	32·14962	32·02226	31·65688	31·12780	30·47745
3·8	32·81818	34·37829	35·11555	35·29902	35·10645	34·65795	34·03614	33·29869
3·9	36·21429	37·88011	38·62415	38·75716	38·48143	37·93182	37·20000	36·34923
4·0	40·00002	41·76595	42·50000	42·56138	42·18020	41·50769	40·64530	39·66217
4·1	44·23684	46·09165	46·79249	46·75499	46·24069	45·41884	44·40133	43·26338
4·2	49·00000	50·92418	51·56000	51·38852	50·70667	49·70330	48·50130	47·18203
4·3	54·38235	56·34460	56·87196	56·52129	55·62888	54·40472	52·98300	51·45104
4·4	60·50000	62·45189	62·81163	62·22322	61·06648	59·57347	57·78962	56·10775
4·5	—	69·36821	69·47964	68·57759	67·08882	65·26786	63·27068	61·19457
4·6	—	77·24602	76·99870	75·68397	73·77764	71·55580	69·18323	66·75995
4·7	—	—	85·52000	83·66259	81·23000	78·51680	75·69329	72·85939
4·8	—	—	95·23182	92·65977	89·56191	86·24448	82·87765	79·55676
4·9	—	—	—	102·85541	98·91311	94·84982	90·82608	86·92588
5·0	—	—	—	114·47304	109·45344	104·46524	99·64417	95·05252
5·1	—	—	—	—	121·39118	115·25000	109·45684	104·03682
5·2	—	—	—	—	134·98447	127·39721	120·41296	113·99636
5·3	—	—	—	—	—	141·14317	132·69118	125·07009
5·4	—	—	—	—	—	156·78005	146·50770	137·42323
5·5	—	—	—	—	—	—	162·12647	151·25376
5·6	—	—	—	—	—	—	179·87296	166·80075
5·7	—	—	—	—	—	—	—	184·35534
5·8	—	—	—	—	—	—	—	204·27568
5·9	—	—	—	—	—	—	—	—
6·0	—	—	—	—	—	—	—	—
6·1	—	—	—	—	—	—	—	—
6·2	—	—	—	—	—	—	—	—
6·3	—	—	—	—	—	—	—	—
6·4	—	—	—	—	—	—	—	—
6·5	—	—	—	—	—	—	—	—
6·6	—	—	—	—	—	—	—	—
6·7	—	—	—	—	—	—	—	—
6·8	—	—	—	—	—	—	—	—
6·9	—	—	—	—	—	—	—	—
7·0	—	—	—	—	—	—	—	—

TABLE I—(*continued*).

Values of β_4.

β_1.

·8	·9	1·0	1·1	1·2	1·3	1·4	1·5	
4·04000	3·92924	3·80711	3·67634	3·53913	3·39727	3·25220	3·10510	1·8
4·63021	4·51000	4·37733	4·23523	4·08613	3·93202	3·77448	3·61479	1·9
5·27513	5·14437	5·00000	4·84537	4·68319	4·51562	4·34441	4·17097	2·0
5·97844	5·83588	5·67849	5·51000	5·33338	5·15101	4·96481	4·77632	2·1
6·74410	6·58831	6·41643	6·23257	6·04000	5·84134	5·63869	5·43372	2·2
7·57640	7·40576	7·21771	7·01679	6·80660	6·59000	6·36928	6·14625	2·3
8·48000	8·29265	8·08654	7·86667	7·63697	7·40059	7·16000	6·91718	2·4
9·45993	9·25376	9·02746	8·78650	8·53521	8·27699	8·01455	7·75000	2·5
10·52168	10·29429	10·04538	9·78094	9·50570	9·22337	8·93686	8·64846	2·6
11·67120	11·41986	11·14560	10·85499	10·55318	10·24419	9·93116	9·61657	2·7
12·91498	12·63657	12·33387	12·01406	11·68273	11·34426	11·68273	10·65861	2·8
14·26010	13·95159	13·61641	13·26400	12·89988	12·52874	12·15424	11·77918	2·9
15·71429	15·37056	15·00000	14·61114	14·21053	13·80322	13·39311	12·98319	3·0
17·28600	16·90296	16·49198	16·06234	15·62111	15·17370	14·72426	14·27595	3·1
18·98452	18·55685	18·10036	17·62506	17·13857	16·64667	16·15375	15·66314	3·2
20·82000	20·34167	19·83386	19·30739	18·77043	18·22912	17·68811	17·15037	3·3
22·80364	22·26770	21·70199	21·11814	20·52486	19·92866	19·33441	18·74572	3·4
24·94775	24·34628	23·71517	23·06690	22·41071	21·75348	21·10026	20·45478	3·5
27·26593	26·58983	25·88479	25·16414	24·43765	23·71249	22·99392	22·28571	3·6
29·77323	29·01036	28·22335	27·42130	26·61616	25·81537	25·02430	24·24676	3·7
32·48632	31·62803	30·74459	29·85091	28·95770	28·07261	27·20107	26·34686	3·8
35·42369	34·45448	33·46360	32·46669	31·47479	30·49567	29·53475	28·59568	3·9
38·60597	37·50988	36·39706	35·28370	34·18110	33·09702	32·03672	31·00367	4·0
42·05617	40·81478	39·56337	37·94426	37·09164	35·89027	34·71942	33·58221	4·1
45·80000	44·39200	42·98291	41·58947	40·22286	38·89031	37·59636	36·34362	4·2 β_2
49·86636	48·26703	46·67828	45·11667	43·59285	42·11346	40·68231	39·30132	4·3
54·28773	52·46836	50·67464	48·92245	47·22155	45·57759	43·99340	42·46993	4·4
59·10079	57·02793	55·00000	53·03157	51·13095	49·30236	47·54727	45·86538	4·5
64·34706	61·98164	59·68567	57·47152	55·34637	53·30941	51·36330	49·50509	4·6
70·07371	67·37000	64·76674	62·27294	59·89174	57·62259	55·46274	53·40809	4·7
76·33454	73·23883	70·28261	67·47004	64·80000	62·26828	59·86897	57·59525	4·8
83·19114	79·64013	76·27770	73·10111	70·10345	67·27568	64·60777	62·08948	4·9
90·71429	86·63316	82·80220	79·20913	75·83926	72·67717	69·70761	66·91597	5·0
98·98572	94·28567	89·91304	85·84250	82·04900	78·50882	75·20000	72·10244	5·1
108·10025	102·67545	97·67502	93·05590	88·77931	84·81081	81·11987	77·67953	5·2
118·16846	111·89225	106·16215	100·91122	96·08264	91·62805	87·50607	83·68109	5·3
129·32001	122·04000	115·45933	109·47890	104·01818	99·01089	94·40185	90·14466	5·4
141·70781	133·23978	125·66434	118·83929	112·65295	107·01593	101·85554	97·11190	5·5
155·51348	145·63333	136·89033	129·08444	122·06304	115·70695	109·92127	104·62922	5·6
170·95414	159·38758	149·26891	140·32027	132·33522	125·15610	118·65982	112·74843	5·7
188·29143	174·70041	162·95392	152·66919	143·56876	135·44525	128·13965	121·52747	5·8
207·84335	191·80793	178·12626	166·27335	155·87777	146·66761	138·43810	131·03138	5·9
—	210·99415	195·00000	181·29870	169·39394	158·92977	149·64286	141·33333	6·0
—	232·60341	213·83011	197·94009	184·27000	172·35402	161·85366	152·51594	6·1
—	—	234·92263	216·42764	200·68400	187·08139	175·18440	164·67274	6·2
—	—	258·64778	237·03500	218·84465	203·27519	189·76568	177·91006	6·3
—	—	—	260·08994	238·99803	221·12555	205·74781	192·34915	6·4
—	—	—	285·98810	261·43629	240·85495	223·30470	208·12889	6·5
—	—	—	—	286·50857	262·72535	242·63850	225·40894	6·6
—	—	—	—	314·63551	287·04710	263·98543	244·37376	6·7
—	—	—	—	—	314·19047	287·62313	265·23750	6·8
—	—	—	—	—	344·60055	313·88000	288·25000	6·9
—	—	—	—	—	—	343·14704	313·70445	7·0

TABLE I—(*continued*).

Values of β_2.

β_1.

β_2	0	·1	·2	·3	·4	·5	·6	·7
1·8	0	1·41563	2·55335	3·45274	4·14938	4·67447	5·05504	5·31428
1·9	0	1·68269	3·03698	4·11024	4·94488	5·57799	6·04158	6·36300
2·0	0	1·99086	3·59423	4·86684	5·85930	6·61567	7·17383	7·56603
2·1	0	2·34609	4·23542	5·73605	6·90840	7·80484	8·47016	8·94237
2·2	0	2·75531	4·97250	6·73343	8·11022	9·16522	9·95138	10·51347
2·3	0	3·29646	5·81933	7·87680	9·48543	10·71931	11·64114	12·30359
2·4	0	3·76936	6·79200	9·18694	11·05781	12·49286	13·56632	14·34023
2·5	0	4·39479	7·90935	10·69172	12·85476	14·51538	15·78319	16·65462
2·6	0	5·11605	9·19345	12·40745	14·90800	16·82084	18·25025	19·28231
2·7	0	5·94878	10·67033	14·37849	17·25435	19·44838	21·08439	22·26384
2·8	0	6·91168	12·37077	16·63931	19·93669	22·44332	24·30640	25·64561
2·9	0	8·02724	14·33140	19·23517	23·00517	25·85827	27·87491	29·48081
3·0	0	9·32265	16·59605	22·21964	26·51868	29·75455	32·13621	33·83063
3·1	0	10·83106	19·21744	25·65651	30·54669	34·20394	36·87766	38·76566
3·2	0	12·59314	22·25945	29·62217	35·17151	39·29040	42·27778	44·36758
3·3	0	14·65928	25·80000	34·20857	40·49118	45·11360	48·43448	50·73124
3·4	0	17·09242	29·93481	39·52721	46·62315	51·79124	55·46278	57·96718
3·5	0	19·97203	34·78258	45·71414	53·70889	59·46367	63·49841	66·20474
3·6	0	23·39961	40·49179	52·93674	61·92000	68·29905	72·70228	75·59586
3·7	0	27·50649	47·25012	61·40265	71·46601	78·50005	83·26613	86·30244
3·8	0	32·46494	55·29753	71·37178	82·60477	90·31249	95·41972	98·58916
3·9	0	38·50483	64·94474	83·17295	95·65633	104·03669	109·44000	112·65722
4·0	0	45·93863	76·60000	97·22686	111·02153	120·04231	125·66296	128·82752
4·1	0	55·20045	90·80847	114·07864	129·20765	138·78825	144·49908	147·46592
4·2	0	66·90894	108·31200	134·44499	150·86399	160·84945	166·45371	169·01630
4·3	0	81·97288	130·14217	159·28370	176·83235	186·95369	192·15427	194·02097
4·4	0	101·77708	157·77045	189·89859	208·21978	218·03271	222·14023	223·14757
4·5	0	128·52933	193·36018	228·10415	246·50617	255·29464	258·14675	257·22494
4·6	—	165·95745	240·21033	276·48965	293·70715	300·32869	300·70801	297·29164
4·7	—	—	303·58400	338·86359	352·62801	355·25957	351·72287	344·66229
4·8	—	—	392·37271	421·03600	427·27315	422·98141	413·36471	401·01993
4·9	—	—	—	532·27910	523·53374	507·52256	488·53923	468·54685
5·0	—	—	—	688·26156	650·39838	614·63369	581·20426	550·11337
5·1	—	—	—	—	822·21327	752·77500	696·86837	649·55652
5·2	—	—	—	—	1063·22283	934·85457	843·39459	772·10198
5·3	—	—	—	—	—	1181·47331	1032·34962	925·02170
5·4	—	—	—	—	—	1527·44262	1281·37852	1118·69384
5·5	—	—	—	—	—	—	1618·63791	1368·38182
5·6	—	—	—	—	—	—	2091·69833	1697·36503
5·7	—	—	—	—	—	—	—	2142·78154
5·8	—	—	—	—	—	—	—	2767·36099
5·9	—	—	—	—	—	—	—	—
6·0	—	—	—	—	—	—	—	—
6·1	—	—	—	—	—	—	—	—
6·2	—	—	—	—	—	—	—	—
6·3	—	—	—	—	—	—	—	—
6·4	—	—	—	—	—	—	—	—
6·5	—	—	—	—	—	—	—	—
6·6	—	—	—	—	—	—	—	—
6·7	—	—	—	—	—	—	—	—
6·8	—	—	—	—	—	—	—	—
6·9	—	—	—	—	—	—	—	—
7·0	—	—	—	—	—	—	—	—

TABLE I—(*continued*).

Values of β_3.

β_1.

·8	·9	1·0	1·1	1·2	1·3	1·4	1·5	
5·47200	5·54505	5·54774	5·49223	5·38882	5·21032	5·07194	4·87218	1·8
6·56556	6·66900	6·69008	6·64297	6·53966	6·39032	6·20353	5·98658	1·9
7·81963	7·95777	8·00000	7·96281	7·86015	7·70376	7·50359	7·26805	2·0
9·25357	9·43082	9·49691	9·47100	9·36924	9·20521	8·99040	8·73446	2·1
10·88912	11·10996	11·20249	11·18895	11·08800	10·91533	10·68412	10·40546	2·2
12·75081	13·01968	13·14100	13·14055	13·03987	12·85700	12·60708	12·30280	2·3
14·86629	15·18749	15·33963	15·35250	15·25093	15·05566	14·78400	14·45047	2·4
17·26684	17·64442	17·82888	17·85464	17·75028	17·53956	17·24229	16·87500	2·5
19·98792	20·42546	20·64305	20·68041	20·57039	20·34018	20·01238	19·60577	2·6
23·06977	23·57017	23·82076	23·86733	23·74757	23·49257	23·12805	22·67532	2·7
26·55821	27·12336	27·40557	27·45754	27·32248	27·03588	26·62694	26·11977	2·8
30·50551	31·13665	31·44672	31·49852	31·34075	31·01385	30·55093	29·97924	2·9
34·97143	35·66579	36·00000	36·04377	35·85359	35·47542	34·94677	34·29832	3·0
40·02452	40·77897	41·12872	41·15378	40·91866	40·47551	39·86670	39·12670	3·1
45·74361	46·55101	46·90489	46·89705	46·60097	46·07576	45·36914	44·51979	3·2
52·21964	53·06850	53·41068	53·35129	52·97396	52·34552	51·51961	50·53845	3·3
59·55782	60·43103	60·73997	60·60490	60·12077	59·36297	58·39163	57·25491	3·4
67·88034	68·75344	69·00044	68·75867	68·13571	67·21640	66·06791	64·74368	3·5
77·32958	78·16863	78·31586	77·92779	77·12604	76·00572	74·64162	73·09286	3·6
88·07212	88·82758	88·82894	88·24432	87·21400	85·84425	84·21798	82·40021	3·7
100·30364	100·91991	100·70481	99·86000	98·53927	96·86080	94·91600	92·77596	3·8
114·25501	114·64607	114·13512	112·94981	111·26193	109·20215	106·87064	104·34450	3·9
130·19998	130·25632	129·34314	127·71610	125·56589	123·03600	120·23522	117·24648	4·0
148·46471	148·04207	146·59001	143·45076	141·66316	138·55442	135·18444	131·64135	4·1
169·44000	168·34874	166·18242	163·23621	159·79886	155·97808	151·91782	147·71022	4·2 β_2.
193·59693	191·58787	188·48207	184·62341	180·25740	175·56124	170·66381	165·65935	4·3
221·50693	218·25251	213·91770	208·87083	203·36995	197·59782	191·68480	185·72423	4·4
253·86810	248·93685	243·00000	236·44133	229·52365	222·42874	215·28302	208·17445	4·5
291·53977	284·36158	276·34066	267·85963	259·17616	250·45088	241·80774	233·31950	4·6
335·58868	325·40700	314·67673	303·75068	292·85347	282·12801	271·66398	261·51581	4·7
387·35126	373·15656	358·90198	344·86280	331·20000	318·00442	305·32303	293·17509	4·8
448·51903	428·95489	410·10784	392·09743	374·96309	358·72156	343·33557	328·77466	4·9
521·25714	494·48636	469·63736	446·54746	425·06230	405·03908	386·34752	368·86975	5·0
608·37258	571·88229	539·15698	509·54711	482·57201	457·86117	435·12000	414·10888	5·1
713·55655	663·87024	620·75355	582·73768	548·81752	518·26993	490·55405	465·25256	5·2
841·74254	773·98556	717·06701	668·15520	625·41017	587·56812	553·72181	523·19655	5·3
999·64805	906·87600	831·47510	768·34849	714·33162	667·33415	625·90596	589·00103	5·4
1196·61837	1068·75185	968·35433	886·54107	818·03932	759·49378	708·65022	663·92692	5·5
1445·98797	1268·06663	1133·45664	1026·85589	939·60867	866·41460	803·82428	749·48239	5·6
1767·36192	1516·57773	1334·46556	1194·63332	1082·92777	991·03145	913·70857	847·48183	5·7
2190·63315	1831·05543	1581·83626	1396·88998	1252·96795	1137·01614	1041·10567	960·12198	5·8
2763·47753	2236·14894	1890·10532	1642·99492	1456·16614	1309·00889	1189·48840	1090·01813	5·9
—	2769·43027	2279·99999	1926·12857	1700·97663	1512·94076	1363·20000	1240·65000	6·0
—	3490·80240	2781·97487	2322·69706	1998·68407	1756·48819	1567·72730	1415·90414	6·1
—	—	3442·43442	2799·25452	2364·63286	2049·72980	1810·08041	1620·93898	6·2
—	—	4335·33729	3412·44751	2820·14136	2406·11294	2099·32960	1862·18916	6·3
—	—	—	4218·75744	3395·58295	2843·91584	2447·37773	2147·87250	6·4
—	—	—	5308·17016	4135·55270	3388·52216	2870·10032	2488·61733	6·5
—	—	—	—	5107·94743	4076·08374	3389·06759	2898·36583	6·6
—	—	—	—	6420·90275	4959·65704	4034·22238	3395·70595	6·7
—	—	—	—	—	6119·99250	4848·19168	4005·88251	6·8
—	—	—	—	—	7685·64544	5893·51196	4763·92498	6·9
—	—	—	—	—	—	7265·32153	5719·68219	7·0

TABLE I—(*continued*).

Values of β_6.

β_1.

β_2.	·0	·1	·2	·3	·4	·5	·6	·7	·8
1·8	9·00000	10·56988	11·37904	11·65931	11·57571	11·24612	10·75489	10·16245	9·5
1·9	11·26008	13·23077	14·24766	14·60515	14·51125	14·11351	13·51686	12·79605	12·0
2·0	14·00000	16·46165	17·72924	18·17685	18·06655	17·58286	16·85605	15·97830	15·0
2·1	17·31490	20·37698	21·94568	22·49677	22·36051	21·76742	20·87902	19·80872	18·6
2·2	21·32037	25·12011	27·00353	27·71253	27·53558	26·80218	25·71216	24·40477	22·9
2·3	26·15756	30·98446	33·20564	34·00193	33·76303	32·84853	31·50583	29·90542	28·1
2·4	32·00000	37·78431	40·64836	41·58212	41·24961	40·10000	38·43902	36·47547	34·3
2·5	39·06250	46·18140	49·64275	50·74739	50·24583	48·78960	46·83136	44·31091	41·7
2·6	47·61300	56·36215	60·52268	61·73207	61·05669	59·19886	56·60720	53·64551	50·4
2·7	55·66909	68·73087	73·70383	75·02652	74·05525	71·66945	68·44455	64·75897	60·8
2·8	70·61765	83·79934	89·70741	91·09765	89·70029	86·61786	82·56030	77·98719	73·1
2·9	86·04990	102·22137	109·19212	110·56655	108·55927	104·55444	98·96957	93·73497	87·8
3·0	105·00000	124·84101	132·99779	134·21520	131·33846	126·10769	119·60133	112·49223	105·2
3·1	128·40949	152·76061	162·70552	163·03622	158·92300	152·05685	143·76373	134·85446	125·8
3·2	157·53846	187·43873	198·22205	198·30084	192·43054	183·37199	172·75506	161·54896	150·3
3·3	194·10417	230·83408	242·90000	241·65332	233·28402	221·27382	207·61816	193·46858	179·4
3·4	240·49650	285·62144	298·71190	295·24380	283·31160	267·30520	249·65531	231·71559	214·2
3·5	300·12500	355·52475	369·00857	361·91859	344·88577	323·43535	300·50441	277·65897	255·7
3·6	378·00027	445·84514	458·40994	445·49756	421·12000	392·20036	362·24145	333·01035	305·3
3·7	481·75409	564·33393	573·41334	551·18736	516·15119	476·89985	437·52021	399·82427	364·9
3·8	623·54548	722·67424	723·36878	686·21072	635·55341	581·87572	529·76495	481·13873	436·6
3·9	823·87500	939·16548	922·09827	860·79199	786·95638	712·91403	643·44000	580·15336	523·3
4·0	1120·00044	1243·83108	1190·70000	1089·74754	980·99234	877·83511	784·43293	701·49500	628·4
4·1	1586·99669	1688·93436	1562·65014	1395·14604	1232·78583	1087·37952	960·60823	851·07448	756·5
4·2	2401·00000	2372·80866	2093·67200	1810·95208	1564·37061	1356·06923	1182·62312	1036·70173	913·6
4·3	4092·27183	3501·41300	2882·34489	2391·55352	2008·75892	1706·85856	1465·15573	1268·83033	1107·2
4·4	9317·00007	5580·42447	4117·65621	3228·42286	2617·09069	2169·64147	1826·36689	1561·65790	1347·6
4·5	—	10228·34035	6204·67651	4485·37892	3471·86981	2792·18750	2303·06656	1934·78712	1648·4
4·6	—	26594·29656	10169·64897	6481·22602	4713·08621	3648·13982	2931·31183	2415·79639	2028·2
4·7	—	—	19470·30398	9912·98937	6594·13318	4857·09743	3779·10320	3044·33255	2512·7
4·8	—	—	56371·22036	16599·18193	9614·97614	6623·62278	4949·12279	3878·84348	3138·1
4·9	—	—	—	32999·08240	14889·79442	9321·73705	6609·16370	5008·09053	3956·7
5·0	—	—	—	107695·38628	25413·19110	13699·35294	9048·42104	6571·75843	5045·8
5·1	—	—	—	—	52445·38896	21455·52501	12800·88518	8799·46863	6523·1
5·2	—	—	—	—	198872·14895	37301·06052	18948·75669	12089·95456	8575·2
5·3	—	—	—	—	—	80097·49338	29999·60559	17186·73944	11510·0
5·4	—	—	—	—	—	372908·59530	53139·45765	25617·41462	15866·6
5·5	—	—	—	—	—	—	119233·63920	40995·34689	22666·2
5·6	—	—	—	—	—	—	759289·60401	74049·27108	34006·3
5·7	—	—	—	—	—	—	—	144676·09326	55018·5
5·8	—	—	—	—	—	—	—	2006409·54750	101473·3
5·9	—	—	—	—	—	—	—	—	253760·7
6·0	—	—	—	—	—	—	—	—	—
6·1	—	—	—	—	—	—	—	—	—
6·2	—	—	—	—	—	—	—	—	—
6·3	—	—	—	—	—	—	—	—	—
6·4	—	—	—	—	—	—	—	—	—
6·5	—	—	—	—	—	—	—	—	—
6·6	—	—	—	—	—	—	—	—	—
6·7	—	—	—	—	—	—	—	—	—
6·8	—	—	—	—	—	—	—	—	—
6·9	—	—	—	—	—	—	—	—	—
7·0	—	—	—	—	—	—	—	—	—

TABLE I—(*continued*).

Values of β_2.

β_1.

·8	·9	1·0	1·1	1·2	1·3	1·4	1·5	
9·51200	8·83433	8·15118	7·47777	6·82451	6·16894	5·60360	5·04289	1·8
12·00421	11·17900	10·34692	9·52638	8·72995	7·96593	7·23956	6·55383	1·9
15·01475	14·01129	13·00000	12·00305	11·03545	10·10708	9·22407	8·38990	2·0
18·63611	17·41688	16·18967	14·98100	13·80869	12·68431	11·61499	10·60464	2·1
22·97697	21·49603	20·00831	18·54532	17·12800	15·76982	14·47888	13·25951	2·2
28·16526	26·36630	24·56380	22·79495	21·08421	19·44700	17·89242	16·42508	2·3
34·35200	32·16574	29·98227	27·84518	25·78270	23·81232	21·94400	20·18247	2·4
41·71577	39·05687	36·41153	33·83049	31·34598	28·97749	26·73564	24·62500	2·5
50·46839	47·23137	44·02511	40·90805	37·91659	35·07200	32·38521	29·86015	2·6
60·86175	56·91597	53·02714	49·26193	45·66075	42·24625	39·02907	36·01182	2·7
73·19633	68·37949	63·65838	59·10810	54·77260	50·67518	46·82519	43·22304	2·8
87·83176	81·94402	76·20351	70·70058	65·47942	60·56257	55·95689	51·65914	2·9
105·20000	97·98379	91·00000	84·33882	78·04766	72·14637	66·63722	61·51151	3·0
125·82186	116·96246	108·44918	100·37681	92·79065	85·70487	79·11430	73·00209	3·1
150·32793	139·42608	129·02981	119·23414	110·07767	101·56439	93·67759	86·38868	3·2
179·48509	166·03608	153·31496	141·40973	130·34509	120·10836	110·66555	101·96895	3·3
214·23055	197·59354	181·99321	167·49870	154·11009	141·78858	130·47480	120·09977	3·4
255·71562	235·07313	215·89514	198·21347	181·98750	167·13881	153·57131	141·18296	3·5
305·36277	279·66656	256·02699	234·41002	214·71069	196·79148	180·50386	165·70000	3·6
364·94018	332·82366	303·61339	277·12110	253·15751	231·49838	211·92066	194·21319	3·7
436·66063	396·40001	360·15227	327·59804	298·38281	272·15609	248·58964	227·38490	3·8
523·31347	472·60352	427·48564	387·36420	351·65936	319·83784	291·42344	265·99719	3·9
628·44297	564·27028	507·89191	458·28341	414·52944	375·83325	341·51036	310·97634	4·0
756·59211	674·95976	604·20700	538·52748	488·87120	441·69836	400·15311	363·42307	4·1
913·64000	809·19917	719·98521	643·29726	576·98306	519·31919	468·91698	424·64982	4·2
1107·27652	972·79761	859·71417	763·76288	681·69350	610·99251	549·69077	496·22740	4·3
1347·67763	1173·28180	1029·10666	908·47666	806·50403	719·52993	644·76391	580·04326	4·4
1648·49012	1420·50928	1235·50000	1083·03746	955·77730	848·39240	756·92502	678·37500	4·5
2028·28450	1727·54471	1488·41118	1294·57784	1135·00321	1001·86621	889·58832	793·98353	4·6
2512·75884	2111·93585	1800·31916	1552·26683	1351·05386	1185·29254	1046·95774	930·23215	4·7
3138·15849	2597·60657	2187·78540	1868·00882	1612·80000	1405·37863	1234·24080	1091·23937	4·8
3956·74047	3217·73134	2673·08744	2257·43040	1931·57581	1670·60729	1457·93113	1282·07756	4·9
5045·80000	4019·21171	3286·65035	2741·29638	2322·13115	1991·80039	1726·18369	1509·03180	5·0
6523·18412	5069·85289	4070·74603	3347·57848	2803·84815	2382·89098	2049·32000	1779·94211	5·1
8575·22779	6470·26979	5085·27254	4114·54025	3402·51244	2862·00092	2440·51504	2104·65868	5·2
11510·03758	8374·45168	6417·05252	5095·44479	4152·90618	3452·96676	2916·74377	2495·65464	5·3
15866·64876	11027·01600	8195·32568	6365·92882	5102·68429	4187·53643	3500·10258	2968·86115	5·4
22666·51071	14834·80538	10618·63416	8035·91250	6318·30392	5108·59113	4219·68376	3544·81789	5·5
34006·36363	20514·98864	14003·83814	10269·52127	7894·34108	6274·97282	5114·27804	4250·28332	5·6
55018·58014	29432·15808	18881·04131	13319·83857	9968·58982	7768·88564	6236·34746	5120·52074	5·7
101473·89624	44471·78151	26191·91258	17592·66245	12747·42898	9707·56565	7657·98899	6202·60000	5·8
253760·75389	72771·12146	37747·20230	23770·94842	16550·29979	12262·25509	9480·10426	7559·91373	5·9
—	137288·89025	57434·99984	32803·61819	21891·83037	15690·22476	11846·90000	9281·23336	6·0
—	368137·95820	95110·70470	47889·60904	29643·43991	20391·20888	14969·55326	11488·51177	6·1
—	—	183958·54967	73394·60360	41378·15038	27012·30625	19166·32023	14358·25475	6·2
—	—	537373·87106	123089·42788	60189·03145	36656·13405	24933·62294	18149·09652	6·3
—	—	—	244764·82983	92941·09117	51329·91458	33079·04770	23252·05212	6·4
—	—	—	796873·64937	158003·73264	75027·70957	44987·59385	30279·45386	6·5
—	—	—	—	324150·69600	116774·84051	63202·70610	40232·52177	6·6
—	—	—	—	1217690·26020	201460·19097	92848·88108	54839·83220	6·7
—	—	—	—	—	428250·86517	145728·18199	77304·82418	6·8
—	—	—	—	—	1965948·20210	255463·29045	114166·67456	6·9
—	—	—	—	—	—	565740·80342	180792·85424	7·0

TABLE II.

Ratio of the Probable Error of the Mode to the Probable Error of the Mean.

β_1.

β_2.	0	·1	·2	·3	·4	·5	·6	·7
1·8	∞	21·3313	4·2009	2·1183	1·2174	·7269	·4193	·2072
1·9	7·7368	457·8417	12·9077	4·3927	2·1953	1·2549	·7466	·4311
2·0	4·3012	33·2972	169·7287	12·4767	4·4553	2·2432	1·2838	·7642
2·1	3·1616	8·4333	84·9902	96·6366	11·7814	4·4488	2·2717	1·3064
2·2	2·5970	4·2726	14·0362	242·9030	65·5132	12·0059	4·4041	2·2867
2·3	2·2634	3·1503	6·5493	22·8026	1167·9192	48·8362	10·3622	4·3384
2·4	2·0460	2·6578	4·2238	9·1310	36·8018	∞	38·6385	9·7353
2·5	1·8957	2·2921	3·1915	5·4143	12·6243	61·2427	1517·6874	31·8420
2·6	1·7881	2·0583	2·6332	3·8779	6·9579	17·4427	108·2317	434·5838
2·7	1·7095	1·8984	2·2922	3·0755	4·7362	8·9118	24·2562	216·1278
2·8	1·6516	1·7842	2·0664	2·6007	3·6263	5·7946	11·4150	34·2005
2·9	1·6103	1·7004	1·9086	2·2942	2·9870	4·2963	7·1814	14·6528
3·0	1·5811	1·6381	1·7942	2·0841	2·5828	3·4550	5·1003	8·6834
3·1	1·5622	1·5918	1·7093	1·9340	2·3105	2·9332	4·0108	6·0586
3·2	1·5520	1·5578	1·6455	1·8238	2·1187	2·5871	3·3476	4·6629
3·3	1·5493	1·5337	1·5977	1·7417	1·9797	2·3467	2·9140	3·8294
3·4	1·5534	1·5180	1·5625	1·6806	1·8776	2·1746	2·6166	3·2922
3·5	1·5637	1·5094	1·5378	1·6359	1·8026	2·0498	2·4061	2·9282
3·6	1·5799	1·5073	1·5223	1·6051	1·7490	1·9595	2·2550	2·6725
3·7	1·6018	1·5114	1·5161	1·5866	1·7133	1·8963	2·1471	2·4899
3·8	1·6293	1·5220	1·5178	1·5803	1·6938	1·8556	2·0726	2·3622
3·9	1·6624	1·5397	1·5301	1·5867	1·6902	1·8351	2·0257	2·2740
4·0	1·7014	1·5665	1·5549	1·6080	1·7033	1·8343	2·0032	2·2188
4·1	1·7464	1·6062	1·5967	1·6477	1·7358	1·8541	2·0041	2·1924
4·2	1·7978	1·6667	1·6633	1·7120	1·7917	1·8969	2·0289	2·1931
4·3	1·8561	1·7653	1·7687	1·8106	1·8777	1·9669	2·0798	2·2207
4·4	1·9221	1·9436	1·9381	1·9588	2·0034	2·0704	2·1558	2·2770
4·5	1·9965	2·3211	2·2203	2·1819	2·1838	2·2167	2·2820	2·3652
4·6	—	3·4126	2·7193	2·5226	2·4412	2·4189	2·4376	2·4905
4·7	—	—	3·7109	3·0601	2·8113	2·6967	2·6538	2·6602
4·8	—	—	6·4341	3·9632	3·3544	3·0796	2·9428	2·8847
4·9	—	—	—	5·7005	4·1910	3·6148	3·3289	3·1786
5·0	—	—	—	10·6664	5·6242	4·3834	3·8501	3·5624
5·1	—	—	—	—	8·1895	5·5415	4·5669	4·0669
5·2	—	—	—	—	16·5324	7·4444	5·5857	4·7390
5·3	—	—	—	—	—	11·1952	7·1126	5·6557
5·4	—	—	—	—	—	24·9391	9·6439	6·9532
5·5	—	—	—	—	—	—	14·7870	8·9040
5·6	—	—	—	—	—	—	38·3578	12·1682
5·7	—	—	—	—	—	—	—	19·0756
5·8	—	—	—	—	—	—	—	20·9345
5·9	—	—	—	—	—	—	—	—
6·0	—	—	—	—	—	—	—	—
6·1	—	—	—	—	—	—	—	—
6·2	—	—	—	—	—	—	—	—
6·3	—	—	—	—	—	—	—	—
6·4	—	—	—	—	—	—	—	—
6·5	—	—	—	—	—	—	—	—
6·6	—	—	—	—	—	—	—	—
6·7	—	—	—	—	—	—	—	—
6·8	—	—	—	—	—	—	—	—
6·9	—	—	—	—	—	—	—	—
7·0	—	—	—	—	—	—	—	—

TABLE II—(*continued*).

Ratio of the Probable Error of the Mode to the Probable Error of the Mean.

β_1.								
·8	·9	1·0	1·1	1·2	1·3	1·4	1·5	
0	—	—	—	—	—	—	—	1·8
·2137	0	—	—	—	—	—	—	1·9
·4418	·2199	0	—	—	—	—	—	2·0
·7794	·4516	·2257	0	—	—	—	—	2·1
1·3238	·7927	·4608	·2313	0	—	—	—	2·2
2·2924	1·3375	·8044	·4693	·2366	0	—	—	2·3
4·2616	2·2914	1·3481	·8148	·4772	·2417	0	—	2·4
9·2167	4·1795	2·2856	1·3563	·8241	·4846	·2466	0	2·5
27·3963	8·6715	4·0960	2·2766	1·3627	·8324	·4916	·2514	2·6
214·2686	23·4630	8·2243	4·0130	2·2652	1·3675	·8399	·4982	2·7
536·8306	132·4970	20·7275	7·8248	3·9320	2·2521	1·3711	·8468	2·8
49·3159	2365·9521	92·5373	18·5702	7·4670	3·8536	2·2381	1·3737	2·9
18·8973	73·5471	∞	69·7007	16·6666	7·1439	3·7785	2·2233	3·0
10·6407	24·5570	115·3008	2797·9031	55·2575	15·3999	6·8551	3·7067	3·1
7·1985	13·0637	32·2652	195·0629	754·5445	45·4504	14·2059	6·5923	3·2
5·4232	8·5553	16·0888	43·0412	373·6907	360·1882	38·4326	13·1965	3·3
4·3841	6·3070	10·1749	19·9059	58·6048	902·3059	216·7873	33·2041	3·4
3·7238	5·0188	7·3339	12·1171	24·7849	82·0317	3859·0983	147·8862	3·5
3·2808	4·2123	5·7431	8·5285	14·4603	31·1179	119·2455	∞	3·6
2·9731	3·6767	4·7652	6·5690	9·9220	17·3081	39·4906	182·8220	3·7
2·7556	3·3080	4·1186	5·3807	7·5111	11·5534	20·7994	50·8071	3·8
2·6024	3·0487	3·6782	4·6092	6·0748	8·5876	13·4722	25·1232	3·9
2·4975	2·8664	3·3701	4·0854	5·1532	6·8542	9·8206	15·7414	4·0
2·4316	2·7414	3·1538	3·6557	4·5324	5·7559	7·7303	11·2373	4·1
2·3991	2·6619	3·0049	3·4647	4·1019	5·0222	6·4239	8·7171	4·2
2·3972	2·6207	2·9087	3·2876	3·7999	4·5157	5·5588	7·1650	4·3
2·4252	2·6139	2·8560	3·1713	3·5899	4·1606	4·9647	6·1470	4·4
2·4841	2·6398	2·8417	3·1043	3·4497	3·9126	4·5484	5·4516	4·5
2·5767	2·6988	2·8630	3·0798	3·3655	3·7440	4·2566	4·9654	4·6
2·7072	2·7928	2·9196	3·0943	3·3282	3·6396	4·0565	4·6235	4·7
2·8822	2·9256	3·0127	3·1462	3·3334	3·5871	3·9275	4·3865	4·8
3·1106	3·1026	3·1451	3·2360	3·3784	3·5806	3·8567	4·2298	4·9
3·4051	3·3318	3·3217	3·3659	3·4629	3·6166	3·8362	4·1379	5·0
3·7831	3·6239	3·5490	3·5398	3·5883	3·6938	3·8611	4·1008	5·1
4·2700	3·9939	3·8366	3·7633	3·7576	3·8129	3·9289	4·1125	5·2
4·9030	4·4629	4·1971	4·0446	3·9757	3·9760	4·0402	4·1696	5·3
5·7402	5·0710	4·6485	4·3947	4·2495	4·1872	4·1957	4·2711	5·4
6·8793	5·8339	5·2155	4·8287	4·5884	4·4523	4·3988	4·4175	5·5
8·4860	6·8516	5·9343	5·3673	5·0054	4·7796	4·6543	4·6113	5·6
10·9159	8·2317	6·6993	6·0399	5·5178	5·1798	4·9694	4·8565	5·7
15·0461	10·1883	8·0738	6·8654	6·1501	5·6681	5·3533	5·1589	5·8
24·2232	13·1659	9·7223	7·9782	6·9362	6·2646	5·8189	5·5274	5·9
—	18·3208	12·0456	9·4138	7·9259	6·9971	6·3809	5·9709	6·0
—	30·4688	15·6688	11·3534	9·1941	7·9053	7·0692	6·5057	6·1
—	—	22·0403	14·1326	10·8607	9·0461	7·9092	7·1508	6·2
—	—	38·1802	18·4461	13·1310	10·5089	8·9484	7·9325	6·3
—	—	—	26·2704	16·3951	12·4328	10·2534	8·8874	6·4
—	—	—	47·9585	21·5227	15·0620	11·9257	10·0672	6·5
—	—	—	—	31·0936	18·8677	14·1298	11·5483	6·6
—	—	—	—	60·8817	24·9280	17·1544	13·4480	6·7
—	—	—	—	—	36·6168	21·5651	15·9472	6·8
—	—	—	—	—	79·1531	28·6967	19·4168	6·9
—	—	—	—	—	—	42·9814	24·5042	7·0

β_2.

its standard deviation is at least an element of importance. It is possible, of course, that the distribution of the mode may be skew. The arithmetical work has been extremely laborious, but will have been repaid, if it enables statistical workers to obtain rapidly an important measure of accuracy by a minute or two of additional computation.

Chapter III. Graphic Representation of the Ratio of the Probable Error of the Mode to the Probable Error of the Mean by a Surface.

(1) *Contour Lines of the Surface of the Probable Error of the Mode.* It will be seen from Table II that the standard deviation of the mode is usually greater than that of the mean and therefore the mode cannot in general be determined with the same accuracy as the mean; within however a large range of values of β_1, β_2, such as most frequently occur in statistical practice, it can be found with *adequate* accuracy, if the size of the sample be reasonably large. For small samples, not only would our method of approaching the problem be illegitimate, but the mode would have no practical value at all.

It will be noted, however, that in a part of the U-curve area the mode, in this case an "antimode," can be determined with greater accuracy than the mean, and it is easy to see that this must physically be so.

In order to get some idea of the nature of the distribution of accuracy of determination of the mode, we may denote the ratio of the standard deviation of the mode to that of the mean by z, and consider the geometry of the surface:

$$z = f(\beta_1, \beta_2),$$

which our Table II represents.

It is clear that there are two asymptotic planes: the first, $5\beta_2 - 6\beta_1 - 9 = 0$, is the vertical plane through the rectangular line, and the other will arise when $\beta_6 = \infty$ or $8\beta_2 - 15\beta_1 - 36 = 0$. In actual distributions, of course, such a value of β_6 cannot occur, but when we fit a Pearson curve by the first four moments $\beta_6 = \infty$ is possible. The surface asymptotes on both sides to the first plane, and it is sufficient to limit it by the second plane for practical purposes, as values of β_1 and β_2 beyond this are very rare. In order to obtain the contour lines of the surface a double series of sections were drawn: i.e. those for β_1 constant and those for β_2 constant. Additional points on these sections were obtained, by interpolation, where needful. From these sections the points corresponding to a constant value of z were read off and plotted to obtain a given contour. In this manner Diagram I was obtained.

(2) *Minimal Lines.* Considerable attention has been devoted to the points on both sections where the series of tangents to the curves are horizontal, i.e. to the points on the curves where z is a minimum. The locus of these points on the contour diagram may be termed a "minimal line." The minimal line for the β_1-sections may be termed the β_1-minimal line and that for the β_2-sections the β_2-minimal line.

RATIO OF PROBABLE ERROR OF MODE TO PROBABLE ERROR OF MEAN

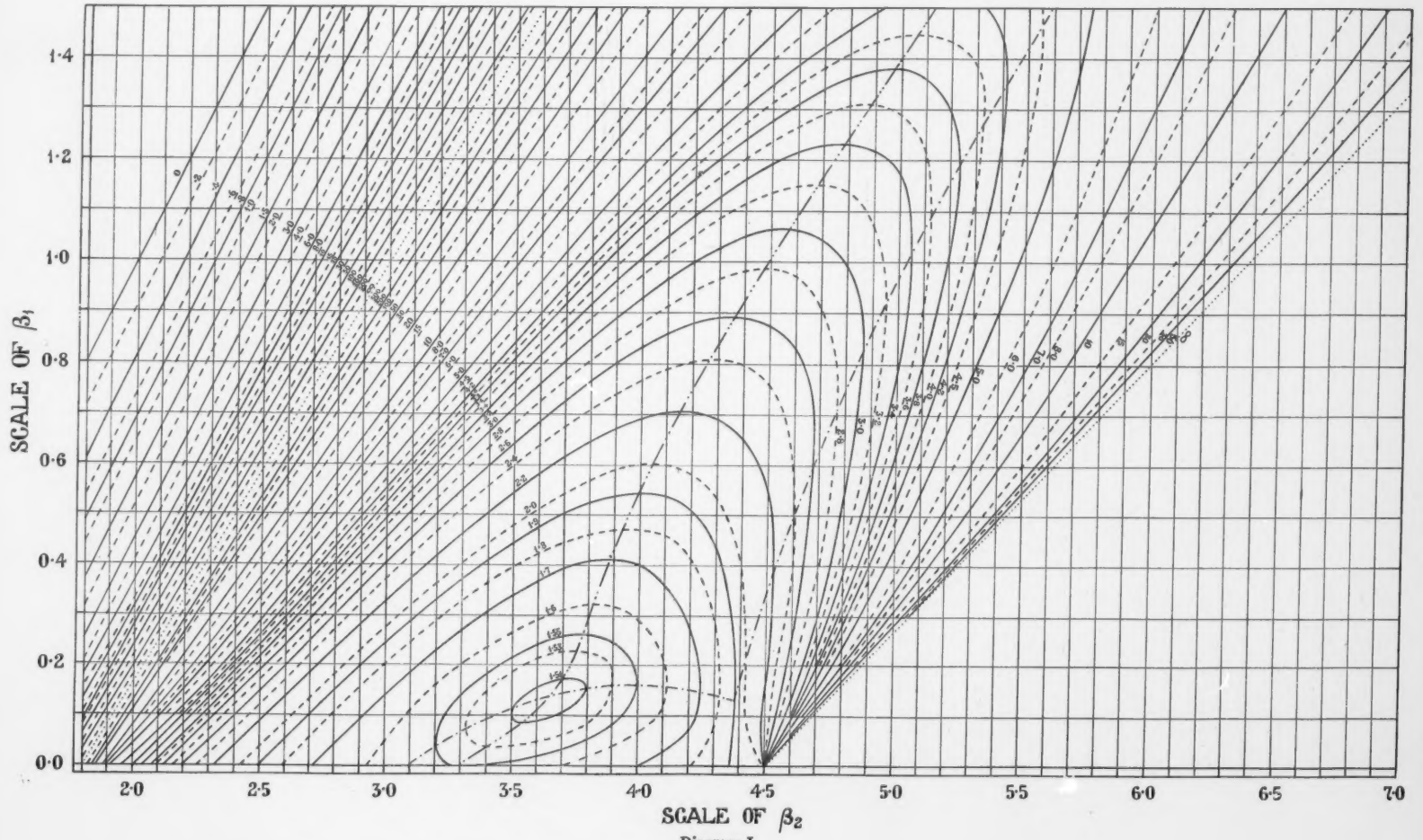


Diagram I.

For determining with adequate accuracy the position of the minimum point of any section, the six-point method in most cases, but in a few the five or four-point method, as given in *Tracts for Computers*, No. II, p. 36, was adopted. A brief description of the method is as follows:

Let the ordinates at six equidistant points x_1, x_2, x_3, x_4, x_5 and x_6 be z_1, z_2, z_3, z_4, z_5 and z_6 respectively, the origin being mid-way between x_3 and x_4.

The abscissa $\bar{x}$ and the ordinate $\bar{z}$ of the minimum point can be found by the following equations:

$$\bar{x}^2\{z_2 - z_1 - 2(z_4 - z_3) + z_6 - z_5\} + \bar{x}\{z_2 - z_1 - 6(z_3 - z_2) - 6(z_4 - z_5) - (z_6 - z_5)\} + \tfrac{1}{12}\{-(z_2 - z_1) + 98(z_4 - z_3) - (z_6 - z_5)\} = 0 \quad \text{.....(31).}$$

Writing this equation $a\bar{x}^2 + b\bar{x} + c = 0$,

we have: $$\bar{z} = \tfrac{1}{2}(z_3 + z_4) - \tfrac{1}{64}b + \tfrac{1}{12}c\bar{x} + \tfrac{1}{48}b\bar{x}^2 \quad \text{.....(32).}$$

The β_1-minimal line is fairly straightforward and the minimum values of z and the corresponding values of β_2 being obtained, the minimal curve can be plotted: see Diagram II. It was clear that this curve could be represented fairly well by a

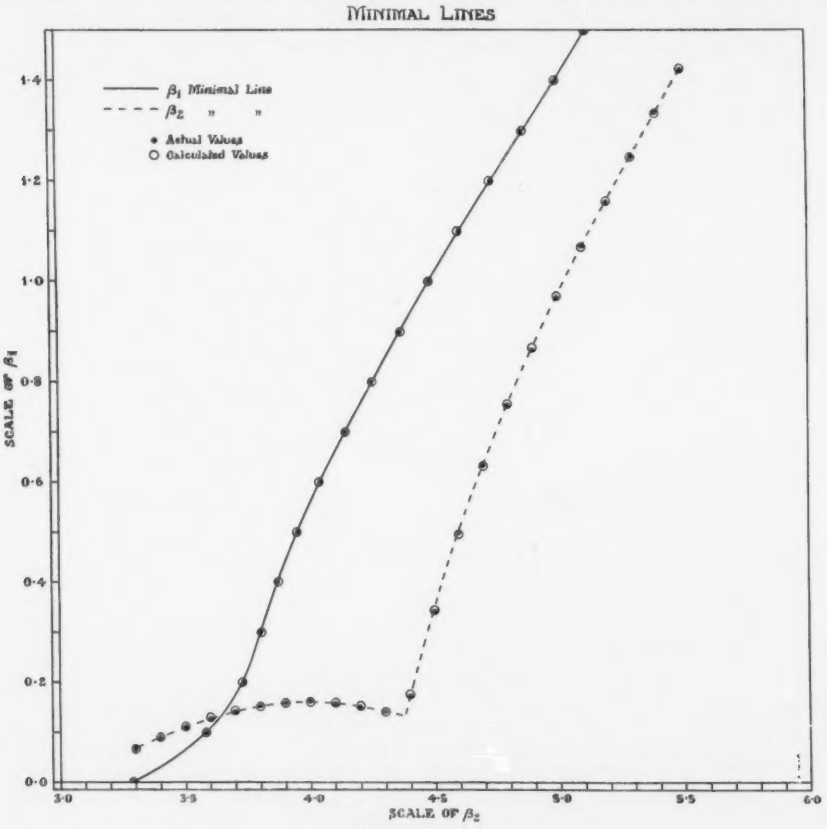


Diagram II.

parabola of the sixth degree. Such a parabola was determined by the method of least squares and its equation was

$$y = 4{\cdot}202293 + {\cdot}05608703x + {\cdot}0005887224x^2 - {\cdot}0001020906x^3 + {\cdot}000006487695x^4 + {\cdot}0000005543824x^5 - {\cdot}00000004007236x^6 \ldots\ldots(33),$$

where y is the calculated value of β_2 and $x = 20\,(\beta_1 - {\cdot}75)$. Table III compares the values of β_2 found from this curve with the plotted values as obtained by computation from Table I.

TABLE III.

β_1	x	Minimum value z	β_2	Calculated by (33)
0	−15	1·5492	3·2884	3·2890
·1	−13	1·5071	3·5833	3·5830
·2	−11	1·5157	3·7318	3·7272
·3	− 9	1·5803	3·8019	3·8082
·4	− 7	1·6896	3·8729	3·8751
·5	− 5	1·8323	3·9541	3·9510
·6	− 3	2·0007	4·0462	4·0424
·7	− 1	2·1894	4·1476	4·1469
·8	1	2·3944	4·2562	4·2589
·9	3	2·6125	4·3707	4·3737
1·0	5	2·8414	4·4892	4·4898
1·1	7	3·0796	4·6119	4·6089
1·2	9	3·3254	4·7371	4·7343
1·3	11	3·5779	4·8647	4·8679
1·4	13	3·8355	5·0043	5·0043
1·5	15	4·0994	5·1249	5·1245

The accordance is within about 1 per cent. and this may be considered reasonable for present purposes. The β_2-minimal line is less simple, because for the portion of the surface dealt with in our diagram it consists of two separate parts, each of which has been fitted by an independent parabola of the third degree. The results are exhibited in Tables IV and V. The intersection of the β_1-minimal and the β_2-minimal curves gives the "bottom of the basin" in the solid surface, and this point has been determined as

$$z = 1{\cdot}505, \quad \text{at} \quad \beta_1 = {\cdot}134, \quad \beta_2 = 3{\cdot}64.$$

This point corresponds on the (β_1, β_2) diagram* to a curve of Pearson's Type IV, a skew curve, and not, as might *à priori* have been supposed, to a normal curve. We see accordingly that for points in the "basin" the mode as determined by the method of moments is always 50 per cent. more inaccurate than the mean, and usually much more than this. Still the mean can be so accurate for large samples that we may learn something from determining the mode even when its probable error is a considerable multiple of that of the mean.

The equation of the parabola fitted to one part of the β_2-minimal line is

$$y = {\cdot}153384 + {\cdot}00831818x - {\cdot}00195078x^2 - {\cdot}00004203768x^3 \quad \ldots(34).$$

* See *Tables for Statisticians*, p. 66.

TABLE IV.

β_2	x	Minimum value z	β_1	y calculated by (34)
3·3	−5	1·5298	·06848	·06828
3·4	−4	1·5177	·09166	·09159
3·5	−3	1·5089	·11141	·11201
3·6	−2	1·5039	·12936	·13078
3·7	−1	1·5041	·14279	·14316
3·8	0	1·5099	·15395	·15338
3·9	1	1·5241	·16023	·15971
4·0	2	1·5495	·16207	·16039
4·1	3	1·5906	·15927	·15965
4·2	4	1·6554	·15181	·15275
4·3	5	1·7598	·14161	·14095

y = computed value of β_1.
$x = 10(\beta_2 - 3{\cdot}8)$.

It will be seen that the deviation of β_1 from its computed value is as a rule less than 1 per cent.

The equation of the parabola fitted to the other part of the β_2-minimal line is

$$y = {\cdot}9205583 + {\cdot}0516564x - {\cdot}00100021x^2 + {\cdot}0000422167x^3 \quad \ldots\ldots(35).$$

TABLE V.

β_2	x	Minimum value z	β_1	y calculated by (35)
4·4	−11	1·9368	·17386	·17512
4·5	− 9	2·1787	·34301	·34386
4·6	− 7	2·4189	·49925	·49547
4·7	− 5	2·6512	·63419	·63199
4·8	− 3	2·8774	·75425	·75545
4·9	− 1	3·0995	·86453	·86786
5·0	1	3·3189	·96775	·97126
5·1	3	3·5362	1·07096	1·06767
5·2	5	3·7524	1·15870	1·15911
5·3	7	3·9678	1·24915	1·24762
5·4	9	4·1825	1·33688	1·33523
5·5	11	4·3971	1·42205	1·42395

y = computed value of β_1.
$x = 10(\beta_2 - 4{\cdot}95)$.

(3) *Model of the Surface.* The surface was modelled by cutting out the contours and building them up at the appropriate values of z. It is seen to be of a very striking and suggestive kind. Photographs of it are provided in Plates I and II.

The reader will find it desirable to compare the contour diagram and the surface. He will then notice that the surface consists essentially of a first portion which we have termed the "basin." This, after a fairly flat portion, rises abruptly to the two asymptotic planes. The asymptotic plane to the right of the figures in Plate I, but to the left in Plate II, is that which bounds the area where the

moments of Type IV curve may become infinite and which is marked on the (β_1, β_2) diagram as the heterotypic area. The boundary of the "basin," on the left of the photographs on Plate I and on the right of the photographs on Plate II, is the vertical plane through the "rectangular line" and beyond this we have another portion of the surface of a scroll-like form which sinks below the bottom of the "basin" and covers the types of curves which may be found above the "rectangular line," including the U- and some J-curves. The portion of the surface shown is limited by three considerations: (1) all real values are limited by $\beta_1 = 0$, hence the surface is not shown below this plane; again (2) the tables have been limited by $\beta_1 = 1{\cdot}5$. which cuts off the back part of the "basin" as shown in Plate I; unlike the front part of the "basin," the value here of the Probable Error of the Mode would be real but some limit had to be given to the tables and the model; (3) a similar limit to the tables is given by $\beta_2 = 7{\cdot}0$. On the "scroll" side a limitation has been made at $\beta_2 = 1{\cdot}8$, although it would have been more proper to have carried it to $\beta_2 = 1$, and limited it by the straight line $\beta_2 - \beta_1 - 1 = 0$. This is the contour line marked 0 in Diagram I of which however only a portion is shown. The contour lines on this portion of the scroll are practically straight lines and for the purposes of the model it was desirable to limit the size. (4) For values lying between $\beta_2 = 1$ and $1{\cdot}8$, and the corresponding value of β_1 falling below the line $\beta_2 - \beta_1 - 1 = 0$, the contour lines of Diagram I are so approximately straight that for practical purposes the value of the probable error may be found by producing them.

CHAPTER IV. STATISTICAL APPLICATIONS.

I propose now to consider by a few examples the application of the above results to determine the probable error of the mode. I shall use several cases of skew distribution, which have already been adopted in earlier researches to illustrate various statistical points. By hatched spaces on the diagrams I shall indicate the relative significance of the probable errors of mode and mean.

Example I. Barometric Height at Laudale (4018 *observations*)*.

Barometric Height	*27·9*	*28·0*	*28·1*	*28·2*	*28·3*	*28·4*	*28·5*	*28·6*	*28·7*	*28·8*
Frequency	1	1	0	0	·5	2·5	2	4·5	8·5	12
Barometric Height	*28·9*	*29·0*	*29·1*	*29·2*	*29·3*	*29·4*	*29·5*	*29·6*	*29·7*	*29·8*
Frequency	32·5	47·5	72·0	87·5	125·5	187·5	255·5	304	346·5	394
Barometric Height	*29·9*	*30·0*	*30·1*	*30·2*	*30·3*	*30·4*	*30·5*	*30·6*	*30·7*	*30·8*
Frequency	380·5	400	361	376	266·5	182	94	50·5	21	2

* *Phil. Trans.* Vol. 190 (1897), pp. 423—469.

The moments were calculated using Sheppard's Corrections. We find:

$$\begin{aligned} \text{Mean} &= 29{\cdot}85699, & \beta_1 &= {\cdot}2039448,\\ \mu_2 &= 14{\cdot}78622, & \beta_2 &= 3{\cdot}200531,\\ \mu_3 &= -25{\cdot}67687, & \sigma &= {\cdot}3845285.\\ \mu_4 &= 699{\cdot}7395, \end{aligned}$$

The distance between the mean and mode:

$$d = {\cdot}093161,$$

and

$$\begin{aligned} \text{Mode} &= 29{\cdot}95015,\\ \kappa &= -{\cdot}7629264, \quad \text{so Type I curve was used,}\\ r &= 56{\cdot}83636, \qquad \epsilon = 458{\cdot}09798,\\ b &= 77{\cdot}65624,\\ m_1 &= 46{\cdot}11299, \qquad m_2 = 8{\cdot}72339,\\ a_1 &= 65{\cdot}30266, \qquad a_2 = 12{\cdot}35358,\\ y_0 &= 421{\cdot}22. \end{aligned}$$

The equation to the curve is

$$y = 421{\cdot}22\left(1+\frac{x}{65{\cdot}3027}\right)^{46{\cdot}1130}\left(1-\frac{x}{12{\cdot}3536}\right)^{8{\cdot}7234}.$$

In finding the ratio of the probable error of the mode to the probable error of the mean the interpolation was made from Table II, using the Mid-Panel Central Difference Formula in *Tracts for Computers*, No. III, p. 7. Let the interpolate divide the square formed by the four nearest interpolants in the x-ratio θ ($\phi = 1-\theta$), and the y-ratio χ ($\psi = 1-\chi$), then

$$\begin{aligned} z_{\theta,\chi} &= \phi\psi z_{0,0} + \phi\chi z_{0,1} + \theta\psi z_{1,0} + \theta\chi z_{1,1}\\ &\quad - \tfrac{1}{6}\theta\phi\{(1+\phi)(\psi\delta^2 z_{0,0} + \chi\delta^2 z_{0,1}) + (1+\theta)(\psi\delta^2 z_{1,0} + \chi\delta^2 z_{1,1})\}\\ &\quad - \tfrac{1}{6}\chi\psi\{(1+\psi)(\phi\delta^{2\prime} z_{0,0} + \theta\delta^{2\prime} z_{1,0}) + (1+\chi)(\phi\delta^{2\prime} z_{0,1} + \theta\delta^{2\prime} z_{1,1})\} \qquad \ldots(36). \end{aligned}$$

	3·1	3·2	3·3	3·4
·1		1·5578 $z_{-1,0}$	1·5337 $z_{-1,1}$	
·2	1·7093 $z_{0,-1}$	1·6455 $z_{0,0}$	1·5977 $z_{0,1}$	1·5625 $z_{0,2}$
·3	1·9340 $z_{1,-1}$	1·8238 $z_{1,0}$	1·7417 $z_{1,1}$	1·6806 $z_{1,2}$
·4		2·1187 $z_{2,0}$	1·9797 $z_{2,1}$	

$$\begin{aligned} \theta &= {\cdot}039448, & \phi &= {\cdot}960552,\\ \chi &= {\cdot}005312, & \psi &= {\cdot}994688.\\ \delta^2 z_{0,0} &= {\cdot}0906, & \delta^{2\prime} z_{0,0} &= {\cdot}0160,\\ \delta^2 z_{1,0} &= {\cdot}1166, & \delta^{2\prime} z_{1,0} &= {\cdot}0281,\\ \delta^2 z_{0,1} &= {\cdot}0800, & \delta^{2\prime} z_{0,1} &= {\cdot}0126,\\ \delta^2 z_{1,1} &= {\cdot}0940, & \delta^{2\prime} z_{1,1} &= {\cdot}0210. \end{aligned}$$

Whence from equation above $z_{\theta,\chi} = 1{\cdot}6503$.

Probable error of mean = ·00410,

" " mode = ·00675.

Mean = 29·8570 ± ·0041,

Mode = 29·9502 ± ·0068.

In the accompanying Diagram III the ranges of *likely* variation of mean and mode are represented by belts of $2\frac{1}{2}$ times the probable error plotted on either side of the mean and mode.

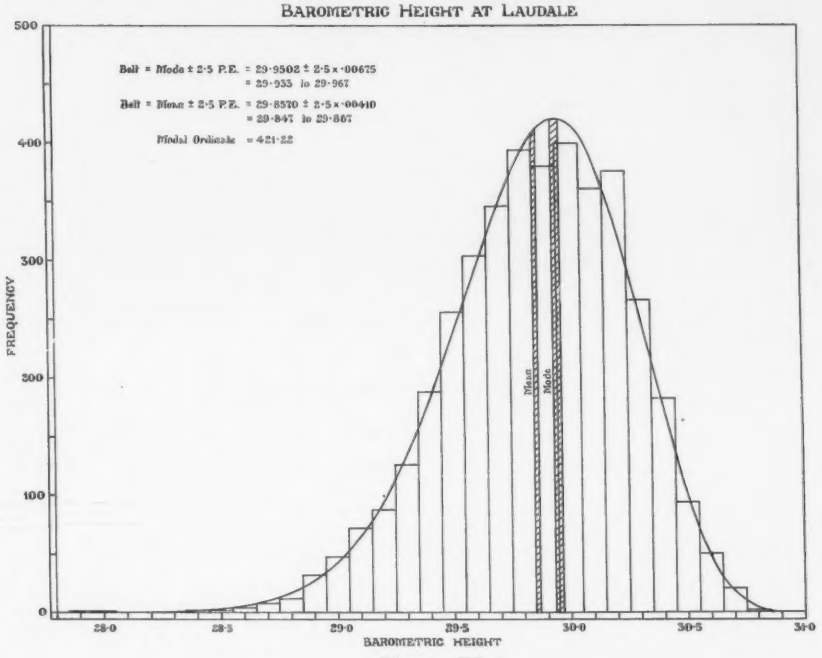


Diagram III.

Example II. Glands of the Fore-Legs of Swine. Right Leg ♀.

In the *Proceedings of the American Academy of Arts and Science*, Vol. XXXII. p. 87, 1896, C. B. Davenport and C. Bullard treated the variation in number of the Müllerian glands in the fore-legs of 4000 swine, and Professor Karl Pearson discussed their material in a memoir in the *Phil. Trans.* Vol. 191 A, pp. 289—296.

The following table gives the distribution for 2000 sows:

Number of Glands	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
Frequency per mille	7·5	104·5	182·5	241	207	138·5	67	36	11	4	1

In computing the moments no correction was made for grouping because this is a case of frequencies concentrated at mid-ordinates. We have

$$\begin{aligned} \text{Mean} &= 3{\cdot}50100, & \beta_1 &= {\cdot}2591775, \\ \mu_2 &= 2{\cdot}824999, & \beta_2 &= 3{\cdot}110825, \\ \mu_3 &= 2{\cdot}417275, & \kappa &= -{\cdot}37311. \\ \mu_4 &= 24{\cdot}82631, \end{aligned}$$

The frequency distribution is of Type I:

$$\begin{aligned} r &= 19{\cdot}98608, & \epsilon &= 72{\cdot}72580, \\ m_1 &= 3{\cdot}78391, & m_2 &= 14{\cdot}20217, \\ a_1 &= 3{\cdot}79632, & a_2 &= 14{\cdot}24873, \\ b &= 18{\cdot}04505, & y_0 &= 237{\cdot}2621, \\ d &= {\cdot}522985, \\ \text{Mode} &= 2{\cdot}97802. \end{aligned}$$

The equation to the curve is

$$y = 237{\cdot}262\left(1+\frac{x}{3{\cdot}7963}\right)^{3{\cdot}7839}\left(1-\frac{x}{14{\cdot}2487}\right)^{14{\cdot}2022}.$$

By an interpolation from my Table II the ratio of the probable errors was found to be 1·8155.

$$\begin{aligned} \text{Probable error of mean} &= {\cdot}67449\sqrt{\frac{2{\cdot}824999}{2000}} \\ &= {\cdot}02535, \\ \text{Probable error of mode} &= {\cdot}02535 \times 1{\cdot}8155 \\ &= {\cdot}04602. \end{aligned}$$

Hence

$$\begin{aligned} \text{Mean} &= 3{\cdot}5010 \pm {\cdot}0253, \\ \text{Mode} &= 2{\cdot}9780 \pm {\cdot}0460. \end{aligned}$$

The result is indicated in Diagram IV. The modal and mean belts or zones are clearly separated and the modal belt sufficiently narrow to be of value.

Example III. Professor Weldon's Crab Measurements No. 4 *.

We will now turn to a nearly symmetrical curve given by the following series of 999 observations:

Index	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
Frequency	1	3	5	11	40	55	98	121	152	147
Index	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19*	*20*
Frequency	126	82	72	41	28	8	7	0	0	2

* *Phil. Trans.* Vol. 185 A (1894), p. 96.

Moments were computed applying Sheppard's Adjustments. I found

$\mu_2 = 7{\cdot}425852$, $\beta_1 = {\cdot}029370$,
$\mu_3 = 3{\cdot}467966$, $\beta_2 = 3{\cdot}137155$,
$\mu_4 = 172{\cdot}9930$, $\kappa = {\cdot}119199$.

Type IV curve fits best.

$r = 67{\cdot}92055$, $m = 34{\cdot}96028$,
$\nu = -24{\cdot}98612$, $a = 22{\cdot}15894$,
$d = {\cdot}2201479$, Origin $= 1{\cdot}533014$.
Mean $= 9{\cdot}684685$, Mode $= 9{\cdot}464537$.
$y_0 = 1{\cdot}76447$, $\sigma = 2{\cdot}72504$.

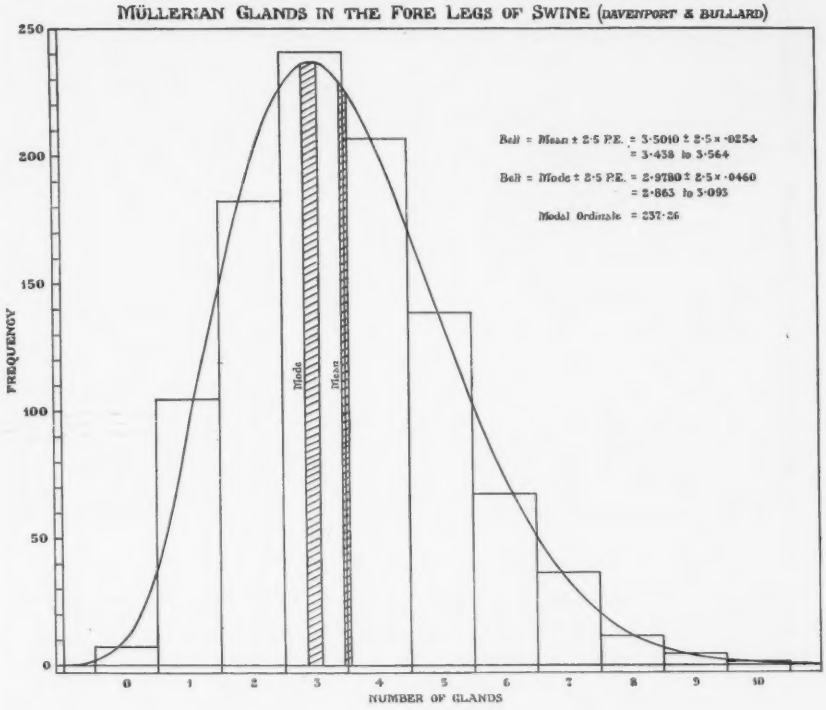


Diagram IV.

The ratio of probable errors was calculated to be 1·5623.

Probable error of mean = ·0582,
" " mode = ·0909.
Mean = 9·6847 ± ·0582,
Mode = 9·4645 ± ·0909.

The equation to the curve is

$$y = 1{\cdot}76447\left(1+\frac{x^2}{(22{\cdot}1589)^2}\right)^{-34{\cdot}9603} e^{24{\cdot}9861\tan^{-1}\frac{x}{22{\cdot}1589}}.$$

This curve was selected because it is very nearly symmetrical, and illustrates a case in which the "modal belt" (see Diagram V) actually overlaps a portion of the "mean belt," and thus we cannot, without further investigation, assert that mean and mode are not identical.

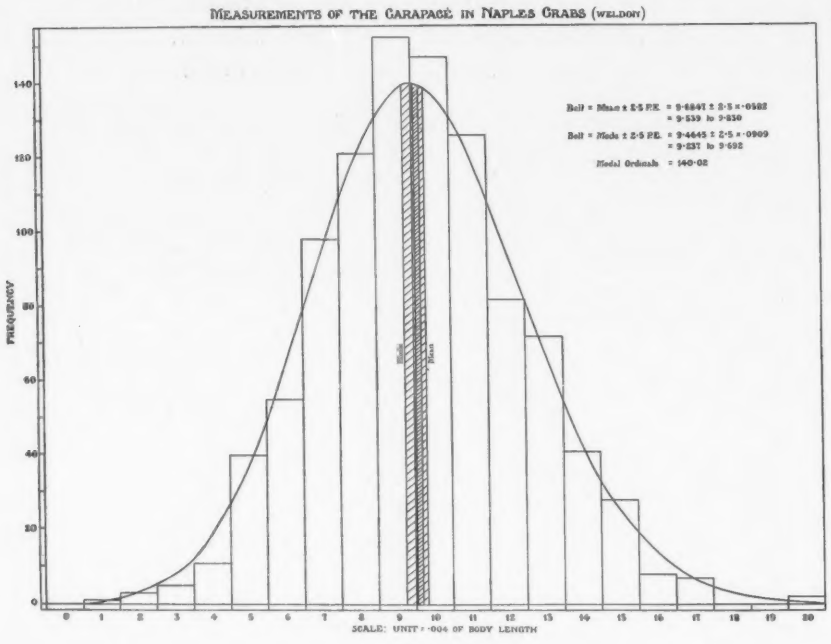


Diagram V.

Example IV. Distribution of 8689 *cases of Enteric Fever received into the Metropolitan Asylums Board Fever Hospitals,* 1871—93.

Age	Number of cases	Age	Number of cases
Under *5*	266	*35—40*	299
5—10	1143	*40—45*	163
10—15	2019	*45—50*	98
15—20	1955	*50—55*	40
20—25	1319	*55—60*	14
25—30	857	Above *60*	13
30—35	503		

I adopted Professor Karl Pearson's redistribution of the tail* dividing the group "Above *60*" as follows:

60—65, 8; *65—70*, 4; *70—75*, 1.

The above frequency distribution has high contact at the old age end but not at the childhood end, thus abruptness corrections are necessary for the calculation of the moments, and for that purpose it becomes desirable to know the sub-divided distribution of the group "Under *5*" into, say, 5 yearly groups.

No data being available for the sub-division for the years 1871—93, the age incidence in the first five years of life provided by the Reports of the Medical Officer of Health for the County of London, for the years 1909—14, which provided 253 cases, were taken, and formed suitable material for adjusting the 266 cases in the above data. They run:

Age ...	*0—1*	*1—2*	*2—3*	*3—4*	*4—5*
Number	14	21	48	78	92

Sub-dividing the group of 266 under 5 proportionally to the above we get:

Age ...	*0—1*	*1—2*	*2—3*	*3—4*	*4—5*
Number	15	22	50	82	97

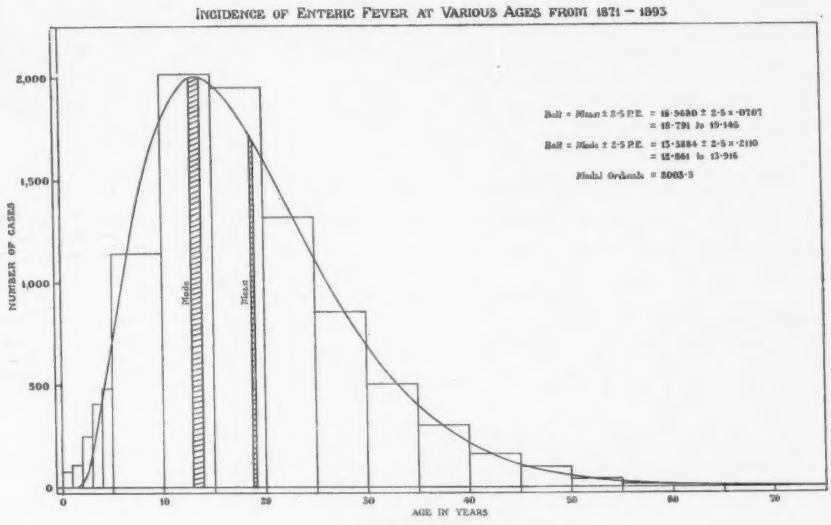


Diagram VI.

The Moments were found, using abruptness corrections† and taking five years as the unit,

$$\mu_2 = 3{\cdot}821776, \qquad \beta_1 = 1{\cdot}033412,$$
$$\mu_3 = 7{\cdot}595118, \qquad \beta_2 = 4{\cdot}348825,$$
$$\mu_4 = 63{\cdot}51882, \qquad \kappa = -2{\cdot}424407.$$

* See *Phil. Trans.* Vol. 186 A (1895), pp. 390—392.

† *Biometrika*, Vol. XII. pp. 231—258.

Type I must be used.

$r = 34{\cdot}50820$, $\epsilon = 108{\cdot}69359$,
$b = 43{\cdot}11413$, $\sigma = 1{\cdot}954936$,
$m_1 = 1{\cdot}736247$, $m_2 = 30{\cdot}77195$,
$a_1 = 2{\cdot}302704$, $a_2 = 40{\cdot}81143$,
$d = 1{\cdot}115930$, $y_0 = 2003{\cdot}526$.
Mean = 18·9680 years, Mode = 13·3884 years.

The equation to the curve is

$$y = 2003{\cdot}53\left(1 + \frac{x}{2{\cdot}3027}\right)^{1{\cdot}7362}\left(1 - \frac{x}{40{\cdot}8114}\right)^{30{\cdot}7720}.$$

The ratio of probable errors was found to be 2·9825.

Probable error of mean = ·014146 in working units,
= ·070728 years,
Probable error of mode = ·21095 years.
Mean = 18·9680 ± ·0707,
Mode = 13·3884 ± ·2109.

The accompanying Diagram VI gives the "likely" belts of variation. It will be seen that mode and mean are quite distinct, and if the modal belt be larger than the mean belt, it still represents quite valuable information as to the likelihood of modal variations.

Example V. Cloudiness at Breslau, 1876—85 (3653 *observations*)*.

We will now take an illustration which indicates how an "antimode" can be more precise than a mean. The following data provide a U-curve.

Degrees of Cloudiness at Breslau.

Degree	0	1	2	3	4	5	6	7	8	9	10
Frequency (days)	751	179	107	69	46	9	21	71	194	117	2089

From the above data the following values result:

$\mu_2 = 18{\cdot}29987$, $\beta_1 = {\cdot}6112252$,
$\mu_3 = -61{\cdot}20297$, $\beta_2 = 1{\cdot}741445$,
$\mu_4 = 583{\cdot}1839$, $\kappa = -{\cdot}1538507$.

Type I is the suitable curve.

$r = {\cdot}1795807$, $\epsilon = {\cdot}0069873$,
$b = 9{\cdot}981426$, $\sigma = 4{\cdot}277834$,
$m_1 = -{\cdot}8774224$, $m_2 = -{\cdot}9429969$.

* *Proc. R. S. London*, Vol. XII. pp. 287—290. Data taken from Hugo Meyer's *Anleitung zur Bearbeitung meteorologischer Beobachtungen für die Klimatologie*, Berlin, 1891, S. 108.

Both m_1 and m_2 being negative, the curve is of U-shape.

a_1	$= 4{\cdot}810939,$	a_2	$= 5{\cdot}170487,$
d	$= 2{\cdot}002148,$	y_0	$= 50{\cdot}74810,$
Mean	$= 6{\cdot}829181,$	Antimode	$= 4{\cdot}827034.$

Hence the equation to the curve is

$$y = 50{\cdot}7481\left(1 + \frac{x}{4{\cdot}8109}\right)^{-{\cdot}8774}\left(1 - \frac{x}{5{\cdot}1705}\right)^{-{\cdot}9430}.$$

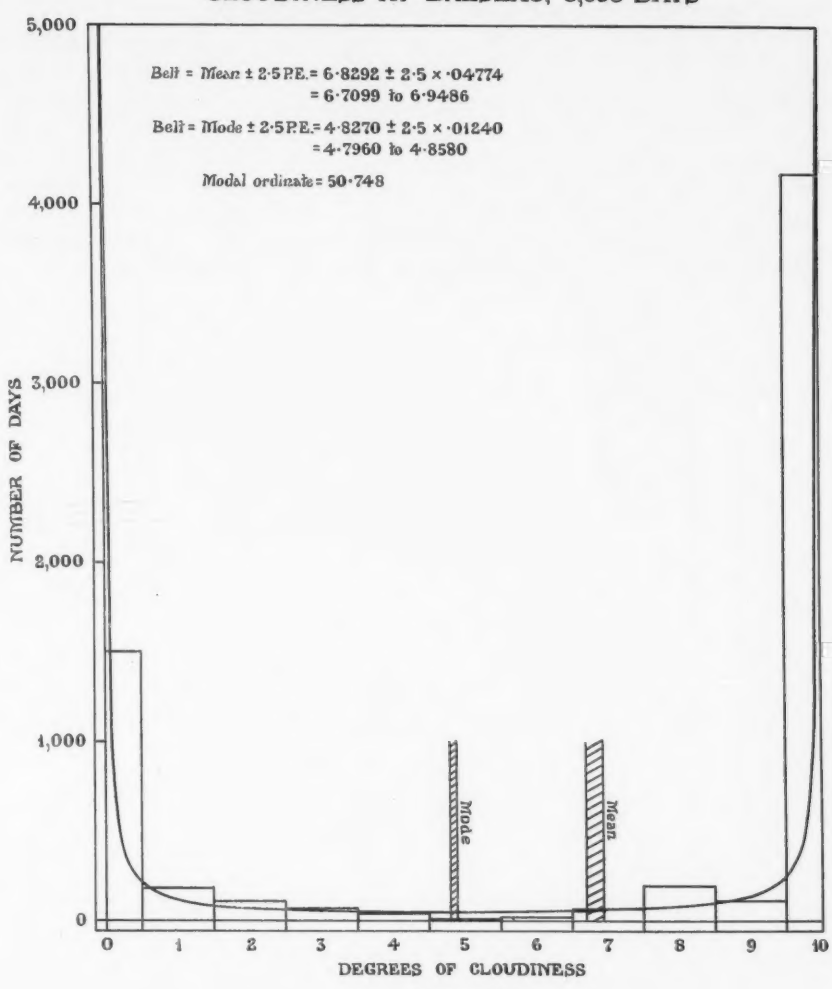


Diagram VII.

The ratio of the probable errors was calculated to be ·25974.

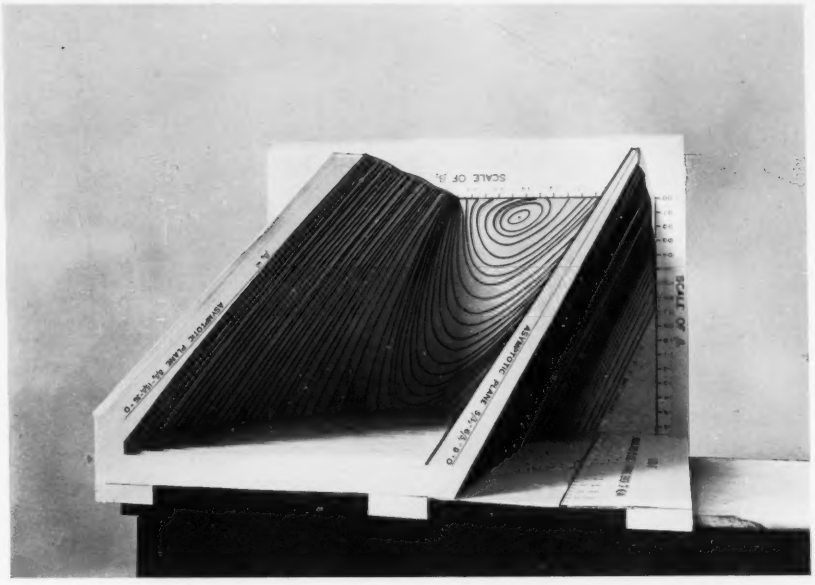
Probable error of mean = ·04774,
" " antimode = ·01240.
Mean = 6·8292 ± ·0477,
Antimode = 4·8270 ± ·0124.

It is clear that the standard deviation of the antimode is only about a quarter of that of the mean, and the ratio in this case corresponds to a value of z on the "scroll" part of the surface (Plate I, A), lying below the bottom of the "basin."

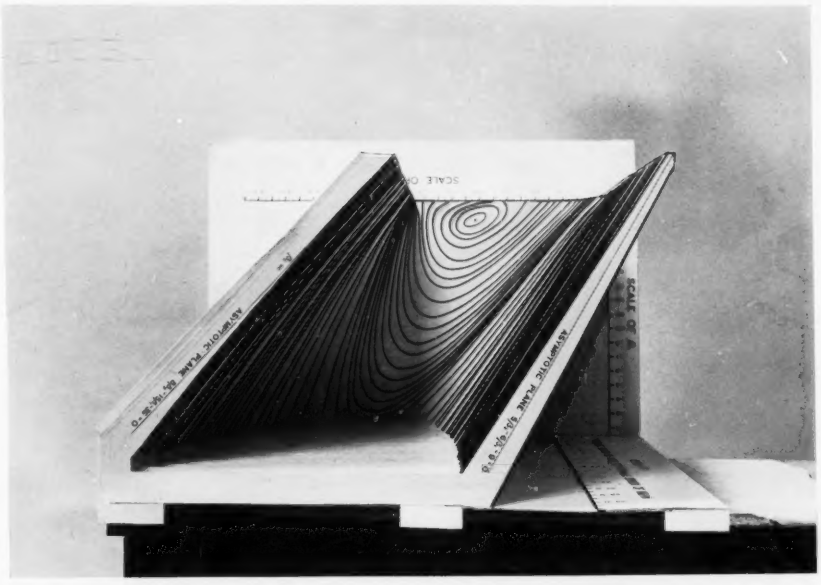
The above give illustrations of the error that is likely to occur in the position of the mode as determined from the four-moment formula. On the corresponding diagrams belts are indicated with ranges of 5 times the probable error round the mean and the mode. It will be seen that while the belt about the mean is very much smaller in most cases than the belt about the mode, yet the latter is sufficiently narrow to show that the mode, as determined by the four-moment formula, has practical significance. In every case such displacements as are suggested as probable by the modal belt are not such as would upset any reasoning which is likely to be based in practice on the observed position of the mode.

Photographs of the surface which gives the Probable Error of the mode, from three aspects. **B** and **C** show the "basin." **A** shows the portion of the surface lying outside the asymptotic plane through the "rectangular line."

D



E



D and **E.** Photographs of the vertical aspect of the surface, showing the "basin" from above and the two asymptotic planes.

UEBER DIE ANWENDUNG DER DIFFERENZENMETHODE ("VARIATE DIFFERENCE METHOD") BEI REIHENAUSGLEICHUNGEN, STABILITÄTSUNTERSUCHUNGEN UND KORRELATIONSMESSUNGEN.

Von OSKAR ANDERSON, Prof. der Polit. Oekon. und Statistik an der Handelshochschule Varna, Bulgarien.

(1) Beim Studium statistischer Reihen und, insbesondere, bei Korrelationsmessungen, kommt es nicht selten vor, daß man sich veranlaßt sieht, die betreffenden Reihen in einzelne Komponenten zu zerlegen. So unterscheidet man bei Reihen, welche die Evolution der Ernteerträge einer gewissen Landschaft darstellen,—eine "säkulare," "evolutorische," oder überhaupt "glatte" Komponente, die den Einfluß des Fortschrittes landwirtschaftlicher Technik wiedergibt, und eine "unregelmäßige," "oszillierende," "zufällige" Komponente, die unter dem Einfluß des Klimas oder der Witterung eines gewissen Jahres steht. Bei "Preiskurven" unterscheidet man oftmals eine "säkulare" und eine wellenartige "Konjunktur-" Komponente, ferner "Saison"-Komponente und eine unregelmäßige "restliche" Komponente, hervorgerufen durch Zufälligkeiten des Börsenspiels, Paniken, Kriege, u.s.w. Es ist eine Anzahl verschiedener "Zerlegungsmethoden" für statistische Reihen vorgeschlagen und mit größerem oder geringerem Erfolge erprobt worden. Als eine derselben kann auch die Differenzenmethode (Variate-Difference-Method) angesehen werden. Das, was letztere, äußerlich, von den übrigen Verfahren unterscheidet, ist der Umstand, daß sie nur bestimmte lineare Funktionen der gesuchten "zufälligen" Reihenkomponente ergeben kann, während erstere die Anmaßung haben, die Reihe in ihre "wahren" Komponenten mehr oder minder genau zu zerlegen. Ob mit Recht—soll im weiteren untersucht werden.

(2) Die bei Zerlegung statistischer Reihen gewöhnlich benutzten Methoden laufen letzten Endes darauf hinaus, das eine oder andere hauptsächlich für Sterblichkeitsmessungen ausgearbeitete "Tafel-Ausgleichungsverfahren" anzuwenden. Ein Charakteristikum dieser Verfahren besteht darin, daß sie immer die "glatte" (wellenartige oder evolutorische) Komponente zu erfassen suchen, das heißt, nur deren "wahrscheinlichsten" oder "vorteilhaftesten" Wert im Bereiche gewisser Fehlergrenzen bestimmen. Die "restliche" Komponente wird dann durch Subtraktion der "ausgeglichenen" von den ursprünglichen Werten der Reihe gewonnen.

Drei Methoden kommen in erster Linie bei solchen "Ausgleichungen" in Betracht*. *Erstens*, kann man eine hypothetisch angenommene oder (in seltenen

* Vergl. E. Czuber: *Wahrscheinlichkeitsrechnung*, 2 Aufl., 1910, II. Band, S. 171.

Fällen) à priori gegebene Funktion der Beobachtungsreihe anzupassen suchen. Gewöhnlich nimmt man an, daß die "glatte" Komponente entweder durch eine ganze rationale Funktion (eine Gerade oder eine Parabel mehr oder minder hoher Ordnung) darstellbar ist, oder aber auch durch eine mehr oder weniger komplizierte trigonometrische Funktion (Fourier Analyse in ihren verschiedenen Formen*). Wird dabei die Zahl der Parameter dieser Funktionen geringer, als diejenige der Reihenglieder, genommen, so müssen die "vorteilhaftesten" Werte der Parameter gefunden werden. Gewöhnlich geschieht das durch Anwendung der Methode der kleinsten Quadrate oder eines ihrer Derivate: Methode der Momente, Methode der Flächen (Cantelli), u.s.w.

Zweitens, kann man "mechanische Ausgleichungsmethoden" gebrauchen. Diese "gehen von der Erwägung aus, daß die *Störung* eines Einzelwertes (das heißt in unserem Falle: der "glatten Komponente") in einer Beobachtungsreihe sich auch noch in den beiderseits benachbarten Gliedern bemerkbar mache, daß jedoch eine sehr geringe Wahrscheinlichkeit dafür bestehe, es werde eine längere Folge von Einzelwerten in gleichem Sinne beeinflußt sein; viel eher sei zu erwarten, daß sich durch den Wechsel des Sinnes auch in kürzeren oder längeren Abschnitten ein Ausgleich im Verlaufe der Erscheinung vollziehe†." Von Wittstein und Woolhouse bis W. F. Sheppard und E. C. Rhodes ist hier eine Anzahl verschiedener Verfahren vorgeschlagen worden, von denen einige die auszugleichende Reihe durch eine Anzahl von ineinandergreifenden Parabeln darzustellen versuchten; in vielen Fällen wurde auch zur Methode der kleinsten Quadrate gegriffen.

Drittens, werden vielfach graphische Ausgleichungsmethoden geübt, wobei "die zusammengehörigen Wertepaare durch Punkte in einem Koordinatensystem dargestellt und nun eine Kurve verzeichnet (wird), die dem Zuge der Punkte möglichst getreu folgt und ihnen 'möglichst nahe' liegt‡." Da ein solches Verfahren, obwohl es in geschickten Händen zu sehr befriedigenden Resultaten führen kann, schließlich nur auf Vertrauen zur Persönlichkeit des Forschers beruht und auch eine wirkliche Bestimmung der möglichen Fehlergrenzen der gefundenen Resultate kaum zuläßt, so werden wir es im weiteren unberücksichtigt lassen.

(3) Wie oben schon angedeutet wurde, beruhen die gebräuchlichen Zerlegungsmethoden statistischer Reihen auf einer *erweiterten Auslegung* der Ausgleichungsverfahren: es wird nämlich stillschweigend angenommen, daß nicht nur die "glatte" Komponente (oder die "glatten" Komponenten) einer Reihe durch die gefundenen "ausgeglichenen" Werte ungefähr darstellbar seien, sondern daß auch die Annäherungswerte der "restlichen" Komponente durch eine gewöhnliche Subtraktion der ausgeglichenen von den ursprünglichen Werten gewonnen werden können. Hierbei wird das relative Größenverhältniss der "glatten" und "restlichen" Elemente zu einander nicht besonders berücksichtigt, obgleich die letzteren gewöhnlich als die beträchtlich kleineren gedacht werden. Unseres Wissens, ist bis jetzt die

* Natürlich, kann man trigonometrische Funktionen auch mit ganzen rationalen kombinieren.

† Czuber, *loc. cit.*, S. 185.

‡ Ebenda, S. 197.

logische Berechtigung eines solchen Verfahrens von niemanden besonders beanstandet worden, obgleich seine Korrektheit, wie wir gleich sehen werden, stark bezweifelt werden könnte.

Es sei, um die der Ausgleichungstheorie eigentümliche Ausdrucksweise zu gebrauchen, je eine Bestimmung der N Größen $G_1, G_2, G_3, \ldots G_N$ gemacht worden, wobei die Werte $u_1, u_2, u_3, \ldots u_N$ gefunden wurden. Die *wahren Fehler* ϵ dieser Bestimmungen sind durch das System (1) gegeben:

$$\left.\begin{aligned} \epsilon_1 &= -u_1 + G_1 \\ \epsilon_2 &= -u_2 + G_2 \\ &\cdots\cdots\cdots\cdots \\ \epsilon_N &= -u_N + G_N \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

G soll hier die "glatte" und $(-\epsilon)$ die *wahre* "restliche" Komponente bedeuten. Auf eine, fürs erste nicht genauer zu definierende, Weise sind nun die "angenäherten ausgeglichenen" Werte von $G_1, G_2, \ldots G_N$ bestimmt worden. Es seien das die Größen $g_1, g_2, \ldots g_N$. Dann sind die *scheinbaren Fehler* λ dieser Größen durch folgendes System ausgedrückt:

$$\left.\begin{aligned} \lambda_1 &= -u_1 + g_1 \\ \lambda_2 &= -u_2 + g_2 \\ &\cdots\cdots\cdots\cdots \\ \lambda_N &= -u_N + g_N \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Eliminiert man u aus beiden Gleichungssystemen, so erhält man unmittelbar:

$$\left.\begin{aligned} \epsilon_1 &= \lambda_1 + G_1 - g_1 \\ \epsilon_2 &= \lambda_2 + G_2 - g_2 \\ &\cdots\cdots\cdots\cdots \\ \epsilon_N &= \lambda_N + G_N - g_N \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Damit also die scheinbaren Fehler den uns einzig interessierenden wahren Fehlern gleich seien, wie es gewöhnlich angenommen wird, müssen alle $G_i - g_i$ von $i = 1$ bis $i = N$ verschwinden.

Nun sei aber folgendes festgestellt. Von seltenen Ausnahmen abgesehen können die "ausgeglichenen" Werte $g_1, g_2, \ldots g_N$ nur aus der Reihe $u_1, u_2, \ldots u_N$ berechnet werden, und zwar bestimmen die meisten Ausgleichungsmethoden, mit gutem Grund, den "ausgeglichenen" Wert g_i als eine *lineare Funktion* F entweder *aller* (klassische Form der Methode der kleinsten Quadrate) oder nur *einiger* (mechanische Ausgleichung) Werte der beobachteten Reihe u. Nehmen wir an, k sei die Anzahl der für die Bestimmung eines jeden g_i benutzten Glieder der Reihe u, so können wir auch abgekürzt schreiben: $g_i = F(u_{i,k})$. Hierbei kann k, je nach der benutzten Methode, eine beliebige positive ganze Zahl zwischen 1 und N bedeuten. Ist nun $F(u_{i,k})$ linear, so hat man mit Berücksichtigung von System (1):

$$g_i = F(u_{i,k}) = F[(G_i - \epsilon_i), k] = F(G_{i,k}) - F(\epsilon_{i,k}).$$

Und System (3) verwandelt sich dann in:

$$\left.\begin{aligned}\lambda_1 &= \epsilon_1 - F(\epsilon_{1,k}) - [G_1 - F(G_{1,k})]\\ \lambda_2 &= \epsilon_2 - F(\epsilon_{2,k}) - [G_2 - F(G_{2,k})]\\ &\cdots\cdots\cdots\cdots\cdots\cdots\\ \lambda_N &= \epsilon_N - F(\epsilon_{N,k}) - [G_N - F(G_{N,k})]\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Damit nun ein beliebiges λ_i dem *wahren Fehler* ϵ_i gleich sei, muß zwangsläufig bei beliebigem i die Bedingung

$$F(\epsilon_{i,k}) = F(G_{i,k}) - G_i$$

erfüllt sein. Dieselbe führt aber entweder zu der sinnwidrigen Annahme, daß eine lineare Funktion der offenbar als mehr oder weniger "zufällig" angenommenen wahren Fehler ϵ durch eine fast identische lineare Funktion der wahren "glatten" Größen G auszudrücken sei; oder aber muß man zugeben, daß dieselbe Funktion F zu gleicher Zeit für die wahren Fehler genau O und für die Größen G_i ebenso genau dieselben G_i ergibt, was ebenfalls einen nicht weniger sinnwidrigen Zusammenhang zwischen G_i und ϵ_i ergeben würde. *Folglich, können die Gleichungen des Systems* (4) *allenfalls nur im Durchschnitt, nur in mathematischer Erwartung annehmbar sein.* Ist die Ausgleichungsformel richtig deduziert und korrekt angewandt, so kann man noch zugeben, daß allenfalls, *bei größerem* ϵ_i, die Differenz $F(G_{i,k}) - G_i$ im Vergleich zu diesem ϵ_i so klein sei, daß sie praktisch vernachlässigt werden könnte, für die Funktion $F(\epsilon_{i,k})$ ist aber auch das nicht gegeben. Meines Erachtens, ergeben daher die gebräuchlichen Ausgleichungsmethoden *im besten Falle bloß* $\epsilon_1 - F(\epsilon_{1,k})$, $\epsilon_2 - F(\epsilon_{2,k})$, ... $\epsilon_N - F(\epsilon_{N,k})$, *keinesfalls aber die gesuchte Reihe* $\epsilon_1, \epsilon_2, \ldots \epsilon_N$*. Und kann auch, wie es nicht selten möglich ist, bewiesen werden, daß die mathematische Erwartung von $\epsilon_i - F(\epsilon_{i,k})$ gleich der mathematischen Erwartung von ϵ_i sei, so folgt daraus noch durchaus nicht, daß die mathematischen Erwartungen der Quadrate dieser Größen ebenfalls einander wenn auch nur ungefähr gleich seien. Und kann sogar letzteres bewiesen werden, so ergibt sich hieraus noch keinesfalls, daß die mathematische Erwartung des Produktes $[\epsilon_i - F(\epsilon_{i,k})][\epsilon_j - F(\epsilon_{j,k})]$ auch nur angenähert der mathematischen Erwartung des Produktes $\epsilon_i \epsilon_j$ gleich kommen muß.

Wir gelangen also zu folgendem vielleicht etwas sensationell anmutendem aber durchaus logisch zwingendem Schluß: *insofern die üblichen Zerlegungsmethoden statistischer Reihen Funktionen (zum Beispiel, Mittlere Fehler, Momente, Korrelationskoeffizienten, u.s.w.) der berechneten "restlichen Komponenten" (also scheinbarer Fehler)* $\lambda_1, \lambda_2, \ldots \lambda_N$ *direkt für entsprechende Funktionen der wahren "restlichen Komponenten" (wahren Fehler)* $\epsilon_1, \epsilon_2, \ldots \epsilon_N$ *ansahen, sind diese Methoden nicht korrekt und können zu Trugschlüssen führen.* Alle mit ihrer Benutzung

* Die Trennung der Funktion $G_i - F(G_{i,k})$ von der Funktion $\epsilon_i - F(\epsilon_{i,k})$ kann, im Allgemeinen, nur bei linearen und einigen wenigen nicht linearen Formen von F gelingen. Das von Prof. W. M. Persons in seinem "Correlation of Time Series" (*Journ. of the Amer. Stat. Assoc.*, June 1923, p. 713 und ff.) vorgeschlagene Ausgleichungsverfahren ist weder linear, noch liefert es eine für Korrektionsberechnungen handliche Form von $F(\epsilon_{i,k})$. Solange die Fehlergrenzen dieser Methode nicht bestimmt sind, dürfte sie daher nur als *graphischen Ausgleichungsmethoden analog* zu betrachten sein.

vorgenommenen Untersuchungen sind zu revidieren und an den berechneten Koeffizienten Korrektionen anzubringen, die im Wesentlichen durch die entsprechende Form von $F(\epsilon_{i,k})$ bestimmt sind. Inwiefern diese Korrektionen die früher gefundenen Resultate umändern werden, hängt sowohl von der Formel des gebrauchten Ausgleichungsverfahrens, als auch, besonders, von den benutzten Funktionen von λ ab. In einer Reihe von Fällen dürften die früheren Forschungsergebnisse im Wesentlichen bestehen bleiben, in andern ist es dagegen zu befürchten, daß sie eine beträchtliche Modifikation erleiden werden. Besonders bezieht sich letzteres auf feinere Untersuchungen, welche ja nur dann logischen Sinn haben können, wenn die zufälligen Fehlergrenzen der gefundenen Werte im Verhältnis zu diesen klein sind.

(4) Im Rahmen dieser Veröffentlichung vermögen wir, natürlich, nicht, die verschiedenen Ausgleichungsmethoden von unserem Standpunkte aus zu sichten und deren Korrektionen zu bestimmen. Die jetzt folgenden zwei Beispiele dienen daher nur zum besseren Verständnis der bisherigen Ausführungen.

Auf S. 295—296 von E. T. Whittaker's und G. Robinson's *The Calculus of Observations* (London, 1924) finden wir die Sheppard'schen Ausgleichungsformeln für Parabeln bis zur 5. Ordnung und für Anzahl der bei der Ausgleichung benutzten Glieder von 3 bis 21 (das heißt, bis $n = 10$). Nehmen wir an: Ausgleichungsparabel 2$^{\text{ter}}$ Ordnung, $n = 2$. Dann ist, in den Bezeichnungen des zitierten Buches, $(u_0 - u_0')$, die Differenz zwischen der beobachteten Größe u_0 und deren nach der Sheppard'schen Methode ausgeglichenem Werte u_0', durch folgenden Ausdruck gegeben:

$$u_0 - u_0' = u_0 - \tfrac{1}{35}\{17u_0 + 12(u_1 + u_{-1}) - 3(u_2 + u_{-2})\}$$
$$= \tfrac{3}{35}\{+ u_{-2} - 4u_{-1} + 6u_0 - 4u_1 + u_2\} = \tfrac{3}{35}\delta^4 u_0,$$

wenn das Symbol $\delta^4 u_0$ die vierte endliche *zentrale* Differenz der Größe u_0 bedeutet. Wird also angenommen, daß im Ausdruck für den scheinbaren Fehler $-\lambda_0 = u_0 - u_0'$ (s. oben, System (4)), kein "glattes" Element G_0 mehr vorhanden ist, so bedeutet das, daß die Differenz $F(G_{0,k}) - G_0$ praktisch gleich Null zu nehmen ist und daß, folglich,

$$\lambda_0 = \epsilon_0 - F(\epsilon_{0,k}) = -\tfrac{3}{35}\delta^4 \epsilon_0.$$

Es erweist sich also hier, daß λ_0 nicht ϵ_0, sondern einem Bruchteil der negativgenommenen 4$^{\text{ten}}$ endlichen Differenz der wahren Fehler ϵ gleichzusetzen ist.

Ein anderes Beispiel. Es liege eine Beobachtungsreihe $u_1, u_2, u_3, \ldots u_N$, vor, wobei N ungerade ist. Ein jedes u_i bestehe aus einem "glatten" Element G_i und einem "restlichen" η_i, welches durch die Gleichung gegeben sei: $\eta_i = -\epsilon_i = u_i - G_i$. Es sei angenommen, daß die "glatte" Reihe $G_1, G_2, \ldots$ durch eine *Gerade*, deren Konstanten nach der Methode der kleinsten Quadrate bestimmt werden, *genau* auszudrücken sei. Letztere ergebe die Werte: $u_1', u_2', \ldots u_N'$.

Dann erhält man, nach den bekannten Regeln der Methode der kleinsten Quadrate, für $u_i - u_i'$, das heißt für die *scheinbare* "restliche" Komponente, schließlich

folgenden Ausdruck:

$$u_i - u_i' = \eta_i - \left\{ \frac{\sum_{i=1}^{N} \eta_i}{N} + \frac{3(N-2i+1)}{(N-1)N(N+1)} \left[(N-1)(\eta_1 - \eta_N) + (N-3)(\eta_2 - \eta_{N-1}) + \ldots + 1 \cdot \left(\eta_{\frac{N-1}{2}} - \eta_{\frac{N+3}{2}} \right) \right] \right\}.$$

Kann die Reihe $\eta_1, \eta_2, \ldots \eta_N$ als eine Reihe von empirischen Größen, die eine zufällige Variable mit einem konstanten Verteilungsgesetz bei N von einander unabhängigen Versuchen annimmt, angesehen werden, so ist die mathematische Erwartung* von u_i' gleich der mathematischen Erwartung von u_i, die mathematische Erwartung von $(u_i - u_i')$ also gleich Null. Was aber die mathematische Erwartung von $\frac{\sum_{i=1}^{N} (u_i - u_i')^2}{N}$ anbetrifft, so ergibt sich für diese nicht der Wert μ_2, sondern $\frac{N-2}{N}\mu_2$, wenn μ_2 die mathematische Erwartung des Quadrates der Abweichung der Größe η_i von ihrer mathematischen Erwartung bedeutet. Kann ferner die Komponente $G_1, G_2, \ldots G_N$ nicht genau durch eine Gerade ausgedrückt werden oder sind die einzelnen η_i nicht vollkommen von einander unabhängig, so werden in der mathematischen Erwartung von $(u_i - u_i')^2$ alle Produkte $\eta_i \eta_j$ von $i=1$ bis $i=N$ auftreten. Eventuell könnte dann die Korrektion des Ausdruckes $\frac{\sum_{i=1}^{N} (u_i - u_i')^2}{N}$ dieselbe Größenordnung, wie er selbst, erlangen. Geht man vollends von der Berechnung einer Geraden zur Ausgleichung nach Parabeln höherer Ordnung über, so wird, wenn diese Ordnung n ist, die mathematische Erwartung von $\frac{\sum_{i=1}^{N} (u_i - u_i')^2}{N}$ gleich $\frac{N-n-1}{N}\mu_2$*, und die Korrektionsformel für $(u_i - u_i')$ mit jeder höheren Ordnung noch komplizierter. Ihre geringe Handlichkeit und die weiten Grenzen,

* In der vorliegenden Arbeit bediene ich mich durchwegs der in England bis jetzt wenig populären "russischen" Methode der mathematischen Erwartungen. Einige Hinweise auf die Eigenschaften und Vorzüge dieser Methode findet der englische Leser im Artikel meines Lehrers Prof. Al. A. Tchouproff's, "On the Mathematical Expectation of the Moments of Frequency Distributions" (*Biometrika*, Vol. XII. November 1918, pp. 140—142), eine vollständigere Darstellung deren Lehrsätze aber im 2. Kapitel von Prof. L. Bortkiewicz's *Iterationen* (Prof. Dr L. v. Bortkiewicz, *Die Iterationen. Ein Beitrag zur Wahrscheinlichkeitstheorie*, Berlin, 1917, S. 30—69).

Die Beschränktheit des der vorliegenden Arbeit zugewiesenen Raumes, läßt es, selbstverständlich, nicht zu, den Rechenweg, auf welchem die meisten der unten angeführten Formeln abgeleitet wurden, auch nur anzudeuten. Dies durfte aber geschehen, weil deren Bestimmung, mit den wenigsten Ausnahmen, eben kein Problem, sondern nur eine Rechenaufgabe darstellt, die, bei genügender Akkuratesse, von einem jeden erledigt werden kann, der über gewisse Kenntnisse in der Theorie der mathematischen Erwartungen sowie in der Kombinatorik verfügt.

† Dieser Satz ist eigentlich schon von Gauß in seiner *Theoria combinationis*, Art. 38, bewiesen worden. Vergl. Czuber: *Wahrscheinlichkeitsrechnung*, 3. Aufl., 1914, Bd. I. S. 347—349.

in denen sich deren Größe bewegen kann, erschweren die Anwendung der Methode der kleinsten Quadrate hier außerordentlich*.

(5) Fassen wir jetzt dasselbe Problem von einer anderen Seite an. Wir haben eben gesehen, daß zwischen der Differenzenmethode und den verschiedenen Reihen-Ausgleichungsmethoden insofern kein Unterschied besteht, als wie die eine so auch die andern die "restliche" Komponente nicht unmittelbar bestimmen, sondern nur deren gewisse lineare Funktionen ergeben können. Es liegt daher der Gedanke nahe, zu untersuchen, ob die Differenzenmethode nicht als ein regelrechtes Reihen-Ausgleichungsverfahren ausgebaut werden könne†.

Es sei wieder $u_1 = G_1 + \eta_1$, $u_2 = G_2 + \eta_2$, ... $u_N = G_N + \eta_N$, eine statistische Reihe, die aus einem "glatten" Element G und einer "restlichen" Komponente η besteht. Nehmen wir nun an, daß die Reihe G_1, G_2, ... G_N sich genau durch eine Parabel $(2k-1)$-ter Ordnung darstellen läßt. Dann wird in der Reihe der $2k$-ten endlichen Differenzen unserer u-Reihe der letzte Rest der Komponente G verschwunden sein und, folglich, wenn wir δ^{2k} als Symbol für die $2k$-te endliche *zentrale* Differenz einführen, wird

$$\delta^{2k} u_i = \delta^{2k} G_i + \delta^{2k} \eta_i = \delta^{2k} \eta_i .$$

* Wohl der bedeutendste Schüler Prof. Al. A. Tchouproff's, Prof. N. S. Tchetverikoff, hat in einer Reihe von Aufsätzen (siehe die in Moskau erscheinenden russischen Zeitschriften "Fragen der Konjunktur," "Statistische Nachrichten," u.s.w.) die Stabilitätsverhältnisse der russischen Ernten und die Zusammenhänge zwischen Ernte und Kornpreis einer genauen Untersuchung unterworfen. Es dürfte nicht leicht sein, in der statistischen Litteratur anderer Länder gleichwertiges an Scharfsinn, Feinheit der angewandten Methoden und an Fleiß zu finden. Und dennoch befürchte ich sehr, daß die Vorliebe Tchetverikoff's für Parabelanpassung nach der Methode der kleinsten Quadrate ihn in einigen Fällen zu Trugschlüssen geführt hat und daß wenigstens einige von seinen "rätselhaften" Korrelationskoeffizienten eben nur "spurious" sind.

Höher in dieser Hinsicht steht die Arbeit einer anderen Schülerin Tchouproff's, des leider zu früh verstorbenen Fräuleins M. Winogradowa: "Der Alkoholkonsum in Rußland und die Ernte," Petrograd, 1916, erschienen in den *Proceedings (Students Section) of the Econom. Departm. of the Petrograd Polytechnic Institute* (russisch).

Frl. M. Winogradowa gebrauchte erste Differenzen.

† Die Grundidee der von "Student," K. Pearson, E. S. Pearson, E. M. Elderton, B. M. Cave, A. Ritchie-Scott, J. Henderson, mir und andern in der *Biometrika* seit 1914 entwickelten Differenzenmethode nehme ich hier als bekannt an. Prof. Truman L. Kelley (*Statistical Method*, New York, 1923, pp. 272, 273) formuliert sie, nach "Student" und K. Pearson, folgendermaßen: "Given two series, $x_1, x_2, \ldots x_n$ and $y_1, y_2, \ldots y_n$ between which there is an organic correlation, R, and a spurious correlation due to a time or location factor such that the two phenomena together result in an apparent, i.e. an obtained correlation, of r. The problem is to determine R. "Student" (1914) has shown that if

$$\begin{aligned} x_1 &= X_1 + bt_1 + ct_1^2 + dt_1^3 + \text{etc.} \\ x_2 &= X_2 + bt_2 + ct_2^2 + dt_2^3 + \text{etc.} \\ &\text{etc.} \end{aligned}$$

and if

$$\begin{aligned} y_1 &= Y_1 + Bt_1 + Ct_1^2 + Dt_1^3 + \text{etc.} \\ y_2 &= Y_2 + Bt_2 + Ct_2^2 + Dt_2^3 + \text{etc.} \\ &\text{etc.} \end{aligned}$$

in which X_1, X_2, etc., Y_1, Y_2, etc., are independent of time or location, then, if the parabolic equations in t terminate with some power t^n, the correlation r_{XY} is given by the correlation between Δ_n and δ_n, the two series of nth order differences, Δ_1 standing for the measures $(x_1 - x_2)$, $(x_2 - x_3) \ldots (x_{n-1} - x_n)$; Δ_2 for the measures $[(x_1 - x_2) - (x_2 - x_3)]$, $[(x_2 - x_3) - (x_3 - x_4)]$, $\ldots [(x_{n-2} - x_{n-1}) - (x_{n-1} - x_n)]$; and similarly Δ_3 for third order differences; Δ_4 for fourth order differences, etc.; the δ's having comparable meanings in the case of the y-series."

Nehmen wir jetzt an, unser Ziel bestehe darin, eine solche lineare Funktion $F(u_{i,\,2k+1})$ zu finden, daß genau $F(G_{i,\,2k+1}) = G_i$. Da nun

$$u_i - F(u_{i,\,2k+1}) = G_i - F(G_{i,\,2k+1}) + \eta_i - F(\eta_{i,\,2k+1}),$$

so führt obige Annahme zwangsläufig zur Gleichung:

$$u_i - F(u_{i,\,2k+1}) = \eta_i - F(\eta_{i,\,2k+1}),$$

wo im rechten Teil die Komponente G gänzlich fehlt.

Nun haben wir eben gesehen, daß unter den eingangs gemachten Bedingungen gerade die $2k$-ten endlichen Differenzen der Reihe u die von uns hier benötigten Eigenschaften besitzen: (1) lineare Funktionen von u zu sein und (2) keinen Rest der Komponente G mehr aufzuweisen. Das führt uns zum naheliegenden Gedanken, $u_i - F(u_{i,\,2k+1})$ gleich $Z\delta^{2k}u_i$ zu setzen, wo Z einen Multiplikator bedeutet, dessen "vorteilhaftester" Wert noch gefunden werden muß. Aus der gemachten Annahme folgt unmittelbar:

$$F(u_{i,\,2k+1}) = u_i - Z\delta^{2k}u_i.$$

Das wäre der gesuchte Ausdruck für F, den wir auch in folgender Form schreiben können:

$$F(u_{i,\,2k+1}) = G_i + \eta_i - Z\delta^{2k}\eta_i.$$

Es bleibt nur noch, den "vorteilhaftesten" Wert für Z zu bestimmen. In Anlehnung an das bei der Methode der kleinsten Quadrate übliche Verfahren, wollen wir Z der Bedingung: $E\sum_{i=1}^{N}[\eta_i - Z\delta^{2k}\eta_i]^2 = minimum$, unterwerfen, wo das Symbol E die mathematische Erwartung bedeutet. Und daraus:

$$E\sum_{i=1}^{N}\eta_i^2 - 2ZE\sum_{i=1}^{N}\eta_i\,\delta^{2k}\eta_i + Z^2E\sum_{i=1}^{N}(\delta^{2k}\eta_i)^2 = minimum.$$

Wenn man die erste Ableitung davon nach Z gleich O setzt, so erhält man für Z den Ausdruck:

$$Z = \frac{E\sum_{i=1}^{N}\eta_i\,\delta^{2k}\eta_i}{E\sum_{i=1}^{N}(\delta^{2k}\eta_i)^2}.$$

Da die zweite Ableitung positiv ist, so ist der gefundene Wert von Z ein Minimum. Um für Z eine einfache Form zu gewinnen, müssen wir bei der Komponente η gewisse Eigenschaften voraussetzen. Es sei wiederum angenommen, daß die Reihe $\eta_1, \eta_2, \ldots \eta_N$, empirische Größen darstellt, welche eine Variable mit einem konstanten Verteilungsgesetz bei N von einander unabhängigen Versuchen ergeben hat. Die Form dieses Verteilungsgesetzes bleibe aber beliebig. Dann ist $E\eta_1^m = E\eta_2^m = \ldots E\eta_i^m$ bei beliebigem m und ferner $E(\eta_i\eta_j) = E\eta_i \,.\, E\eta_j$, wenn $i \neq j$. Nach einigen Umformungen gelangen wir dann zum Ausdruck:

$$Z = \frac{C_{2k}^{k}}{C_{4k}^{2k}} = \frac{[(2k)!]^3}{(4k)!\;k!\;k!}.$$

Derselbe Ausdruck würde sich auch ergeben, wenn wir als Bedingungsgleichung für Z einfach $E[\eta_i - Z\delta^{2k}\eta_i]^2 = minimum$, nehmen würden. Der von

uns eingeschlagene Weg dürfte aber die Anlehnung an die Methode der kleinsten Quadrate klarer hervortreten lassen.

Setzt man den gefundenen Wert von Z in die Formel für $F(u_{i,2k+1})$ ein, so erhält man die endgiltige Formel des der Differenzenmethode entsprechenden Ausgleichungsverfahrens:

$$F(u_{i,2k+1}) = u_i' = u_i - \frac{[(2k)!]^3}{(4k)!\ k!\ k!}\ \delta^{2k} u_i \quad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Gebraucht man in dieser Formel O, statt i, als Subscriptum, und gibt man k sukzessive die Werte 1, 2, 3, 4, ... so erhält man unmittelbar:

bei $k=1$, $u_0' = \frac{1}{3}[u_0 + (u_1 + u_{-1})]$,
„ $k=2$, $u_0' = \frac{1}{35}[17u_0 + 12(u_1+u_{-1}) - 3(u_2+u_{-2})]$,
„ $k=3$, $u_0' = \frac{1}{231}[131u_0 + 75(u_1+u_{-1}) - 30(u_2+u_{-2}) + 5(u_3+u_{-3})]$,
etc.

Also genau dieselben Koeffizienten, die die Sheppard'sche Ausgleichung in dem Falle ergibt, wenn man sein n der Ordnung seiner Parabel (oder der Hälfte der Ordnung unserer Differenz) gleich setzt! (vergl. Whittaker and Robinson, *loc. cit.* pp. 295—296).

Will man also hier, bei Sheppard, die "restliche Komponente" für einen Annäherungswert der wahren Komponente η ansehen, so darf man der Differenzenmethode auch nicht vorwerfen, daß sie nur lineare Funktionen der Komponente η ergeben könne.

Gibt man, im Gegenteil, zu, daß die nach Sheppard berechnete restliche Komponente $u_i - u_i'$ nicht gleich η_i sei (und wir glauben, daß man gezwungen ist, dies zuzugeben), so muß man die Differenzenmethode, als ein in gewissen Hinsichten ebenbürtiges Mitglied, in die Reihe der anderen Ausgleichungsmethoden aufnehmen. Meines Erachtens, ist eben das letztere geboten.

(6) Um die weiteren Ausführungen der vorliegenden Schrift verständlich zu machen, ist es notwendig ihnen eine Darstellung der wichtigsten Formeln der Differenzenmethode vorausgehen zu lassen. Einige derselben werden hierbei zum ersten Mal der Öffentlichkeit übergeben.

Eine gegebene statistische Reihe $u_1, u_2, \ldots u_N$ bestehe aus einer "glatten" Komponente G und einer "restlichen" s, so daß $u_i = G_i + s_i$. Es sei ferner angenommen, daß durch die k-te endliche Differenz erstere Komponente gänzlich vertilgt werde und daß, folglich, $\Delta^k u_i = \Delta^k G_i + \Delta^k s_i = \Delta^k s_i$*. Wir besitzen dann

* Begnügt man sich damit, daß $\Delta^k G_i$, ohne ganz zu verschwinden, nur einen gewissen, für die Praxis zu vernachlässigenden, Teil von $\Delta^k u_i$ ausmachen könne, so werden, außer den durch eine ganze rationale algebraische Funktion (Parabel) bis $(k-1)$-ter Ordnung darstellbaren Reihen, auch verschiedene andere, durch andere algebraische oder transzendente Funktionen darstellbare Reihen, ungefähr zum selben Resultat führen. So auch Yule's Sinus-Reihen mit der Formel $u_i = A \sin\left(2\pi \frac{t+\tau+ih}{T}\right)$ im Intervall $O \leqq h \leqq \frac{1}{4}T$ und $T \geqq h \geqq \frac{3}{4}T$. (Vergl. den Artikel von Yule im *Journ. Roy. Stat. Soc.*, Vol. LXXXIV. p. 497 et seq.)

Weiter unten werden wir noch die allgemeinen Bedingungsgleichungen feststellen, welche die Eigenschaften der beim endlichen Differenzieren allmählich an Bedeutung verlierenden Reihen der "Gruppe G" umschreiben. (Vergl. § 10.)

$(N-k)$ lineare Funktionen der N Größen $s_1, s_2, \ldots s_N$, von denen eine jede von der Form

$$\Delta^k s_i = s_i - C_k^1 s_{i+1} + C_k^2 s_{i+2} - C_k^3 s_{i+3} + \ldots + (-1)^k C_k^k s_{i+k} \quad \ldots\ldots\ldots(6$$

ist, wenn

$$\left(C_k^j = \frac{k!}{j!\,(k-j)!} \ldots\right)^* \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6a).$$

$N-k$ Gleichungen genügen aber nicht zur Bestimmung von N Unbekannten, und folglich, können die Größen $s_1, s_2, \ldots s_N$, keinesfalls allein aus der Reihe der Funktionen $\Delta^k s$ berechnet werden. Diese komplizierten Funktionen können an und für sich auch durchaus nicht als Repräsentanten oder angenäherte Werte der Reihe s gelten. In dieser Hinsicht haben G. U. Yule und W. M. Person ganz recht†. Sie sollen es aber auch garnicht! Die einzige Bedeutung dieser Funktionen besteht eben darin, daß man aus ihnen, bei Einführung gewisser Hypothesen über die Beschaffenheit der ursprünglichen Reihe s, verschiedene Charakteristiken derselben berechnen kann. *Nimmt man an, letztere könne angesehen werden, als eine Reihe von empirischen Größen, die eine zufällige Variable mit einem konstanten Verteilungsgesetz bei N von einander unabhängigen Versuchen annimmt, so gestalten sich die Formeln für die gesuchten Charakteristiken besonders einfach.*

Als erste dieser Formeln sei der mittlere Fehler (Standard Deviation) σ bestimmt.

Bedeutet das Symbol E wiederum die mathematische Erwartung, und gebraucht man die Bezeichnungen Prof. Al. A. Tchouproff's‡:

$$m_r = Es^r, \quad \mu_r = E(s-m_1)^r \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

so ist, bekanntlich, der mittlere Fehler (Standard Deviation):

$$\sigma = \sqrt{\mu_2} = \sqrt{E(s-m_1)^2} = \sqrt{m_2 - m_1^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(8).$$

* In der Theorie der endlichen Differenzen steht gewöhnlich $\Delta' u_1$ für $u_2 - u_1$ und, überhaupt, $\Delta^k u_i$ für $\Delta^{k-1} u_{i+1} - \Delta^{k-1} u_i$. Das macht wohl die Analogie mit der Differentialrechnung evidenter, die im Texte von uns gebrauchte Bezeichnung ist jedoch für den praktischen Gebrauch entschieden bequemer, da wenn die Reihe in einer vertikalen Kolonne angeordnet ist, man aus dem oberen das ihm unmittelbar folgende untere Glied subtrahiert und nicht umgekehrt. Letzteres könnte eine unnütze Fehlerquelle abgeben. Dieselben praktischen Erwägungen veranlaßten mich auch, vom Gebrauche der sogenannten *zentralen Differenzen* hier abzustehen, obgleich einige Formeln durch Einführung der letzteren ein eleganteres Aussehen erhalten hätten.

An diesem Orte sei noch besonders auf die unbedingte Notwendigkeit *der Kontrollen* bei Berechnung der endl. Differenzen hingewiesen. Am bequemsten ist die Benutzung der Kontrollformel

$$\sum_{i=1}^{N-k} \Delta^k s_i = \Delta^{k-1} s_1 - \Delta^{k-1} s_{N-k+1}.$$

† Es bedarf garnicht der ziemlich weitschweifigen Ausführungen W. M. Person's, um die "theory of the tendency of signs of terms of higher differences to alternate" zu beweisen. (Vergl. sein "On the variate difference correlation method and curve fitting" in *Quarterly Publications of the American Statistical Association*, New Series, No. 118 (Vol. xv.), June 1917, p. 612.) Es leuchtet ohne weiteres ein, daß wenn auch die einzelnen s_i von einander vollkommen unabhängig sind, schon die *erste Differenz* $s_i - s_{i+1}$ mit der ihr folgenden ersten Differenz $s_{i+1} - s_{i+2}$ *negativ* korreliert sein muß, und daß $\Delta^k s_{i+1}$, in welchem k von $k+1$ einzelnen s dieselben sind, wie im Ausdruck für $\Delta^k s_i$, mit diesem desto stärker *negativ* korreliert ist, je größer k genommen wird.

Die Werte dieser Korrelationen sind weiter unten durch Formel (19) gegeben.

‡ *Loc. cit.*, p. 148.

Die Größe $\sigma^2 = \mu_2$ werde ich hier apriorische Streuung (Variance) nennen.

Dem *apriorischen* mittleren Fehler σ sind bei der oben gemachten Annahme die mathematischen Erwartungen zweier Formen des *empirischen* mittleren Fehlers σ' gleich, die wir durch ein Subskriptum 0 von einander unterscheiden wollen:

$$\sigma_0' = \sqrt{\frac{\sum_{i=1}^{N} (s_i - m_1)^2}{N}} \quad \text{.....(9),}$$

und

$$\sigma' = \sqrt{\frac{\sum_{i=1}^{N} (s_i - s_{(N)})^2}{N-1}}, \text{ wenn } s_{(N)} = \frac{\sum_{i=1}^{N} s_i}{N} \quad \text{.....(10).}$$

Es ist also dann $E\sigma_0' = E\sigma' = \sigma = \sqrt{\mu_2}$.

Führt man die Bezeichnung ein:

$$\sigma_k' = \sqrt{\frac{1}{C_{2k}} \cdot \frac{\sum_{i=1}^{N-k} (\Delta^k s_i)^2}{N-k}} \quad \text{.....(11),}$$

wo $\Delta^k s_i$ durch die Formel (6) gegeben ist und C_{2k} (vergl. Form. 6*a*) gleich $\frac{2k!}{k!\,k!}$, so kann es—immer unter derselben Annahme einer zufälligen Variablen mit konstantem Verteilungsgesetz und N unabhängigen Versuchen—ganz strenge bewiesen werden, daß

$$E\sigma_k'^2 = \sigma^2 = \mu_2.$$

Diese Gleichung bleibt hier genau, unbeschadet, ob $N-k$ gleich 1 oder einer Million sei*.

Ist k so groß, daß man für seine Faktorial-Funktionen die Stirling'sche Formel anwenden darf, so kann (11) auch in folgender Form geschrieben werden:

$$\sigma_k'^2 = \frac{\sqrt{\pi k}}{4^k} \cdot \frac{\sum_{i=1}^{N-k} \Delta^k s_i^2}{N-k} \quad \text{.....(11a).}$$

Schon bei $k = 6$ beträgt der Fehler, den man bei Anwendung dieser Annäherungsformel begeht, kaum 2,1 % des wahren Wertes, und bei $k = 9$—bloß 1,4 %.

* Formel (11) dient nur zur Bestimmung der uns einzig interessierenden Größe μ_2. Es hat daher keinen Sinn, ihr eine Form zu verleihen, die der Formel (10) analog wäre, also etwa

$$\frac{1}{C_{2k}(N-k)} \cdot \sum_{i=1}^{N-k} \left(\Delta^k x_i - \frac{1}{N-k} \sum_{i=1}^{N-k} \Delta^k x_i \right)^2.$$

Dadurch wird nur der Rechenweg komplizierter, aber der wahrscheinliche Fehler der Bestimmung nicht kleiner. Ebenso wenig Sinn hat es, zur Bestimmung von $\sigma_k'^2$ nicht gleich alle erhaltbaren Werte von s hinzuzuziehen, sondern, um die Zahl der $\Delta^k s$ immer konstant bleiben zu lassen, mit jeder neuen endlichen Differenz je ein neues Glied s der Reihe hinzuzufügen. Der Gewinn (Vereinfachung der Formeln) ist gering und die mittleren Fehler für die ersten Differenzen werden nur unnützerweise vergrößert. Es ist ziemlich dasselbe, als wenn wir bei Anwendung der Methode der kleinsten Quadrate einen Teil der möglichen Normalgleichungen unberücksichtigt ließen.

Eine ganz andere Frage ist es, natürlich, wie weit die empirische Größe $\sigma_k'^2$ sich in der Praxis von seiner mathematischen Erwartung μ_2 entfernen könne. Zur Lösung dieser Frage ist die Bestimmung des mittleren Fehlers von $\sigma_k'^2$ erforderlich, und hier spielt, selbstredend, die absolute Größe von $N-k$ eine ausschlaggebende Rolle.

$\sigma^2_{(\sigma^2,\,k)}$, die apriorische Streuung (Variance) von $\sigma_k'^2$, bestimmt sich aus der Formel

$$\sigma^2_{(\sigma^2,\,k)} = E\,(\sigma_k'^2 - \mu_2)^2.$$

Nach ziemlich langwierigen Umformungen ergibt sich hieraus (immer nur unter denselben Annahmen über die Natur der Reihe s), *wenn* $k \leqq \frac{N}{2}$:

$$\sigma^2_{(\sigma^2,\,k)} = \frac{\mu_4 - 3\mu_2^2}{N-k}\left[1 - \frac{2S_{(k)}}{(C_{2k}^{\,k})^2\,(N-k)}\right] + \frac{2\mu_2^2}{N-k}\left[\frac{C_{4k}^{\,2k}}{(C_{2k}^{\,k})^2} - \frac{k}{2\,(N-k)}\right] \ldots(12),$$

wobei

$$S_{(k)} = \sum_{i=0}^{k-1}(C_k^{\,i}\,C_k^{\,i+1})^2 + 2\sum_{i=1}^{k-2}(C_k^{\,i}\,C_k^{\,i+2})^2 + 3\sum_{i=1}^{k-3}(C_k^{\,i}\,C_k^{\,i+3})^2 + \ldots + k\,(C_k^{\,0}\,C_k^{\,k})^2 \ldots *.$$

Da es bewiesen werden kann, daß jedenfalls

$$\frac{k}{2\,(N-k)} \geqq \frac{2S_{(k)}}{(C_{2k}^{\,k})^2\,(N-k)},$$

so erhält man, wenn man für größere $(N-k)$ die Ausdrücke von der Ordnung $\frac{1}{(N-k)^3}$ unterdrückt, angenähert:

$$\sigma^2_{(\sigma^2,\,k)} = \frac{\mu_4 - 3\mu_2^2}{N-k} + \frac{C_{4k}^{\,2k}}{(C_{2k}^{\,k})^2}\cdot\frac{2\mu_2^2}{N-k} \quad \ldots\ldots\ldots\ldots\ldots\ldots(13).$$

Ist schließlich k so groß, daß für seine Faktorial-Funktionen die Stirling'sche Formel angewandt zu werden vermag, so kann man auch annäherungsweise schreiben:

$$\sigma^2_{(\sigma^2,\,k)} = \frac{\mu_4 - 3\mu_2^2}{N-k} + \frac{\mu_2^2\sqrt{2k\pi}}{N-k} \quad \ldots\ldots\ldots\ldots\ldots\ldots(13a).$$

Schon bei $k=6$ beträgt der Gesamtfehler bei Anwendung dieser Formel bloß etwa 3 %.

Formeln (12), (13) und (13*a*) gelten für ein beliebiges konstantes Verteilungsgesetz der Variablen. Ist es jedoch "normal," so besteht, bekanntlich, die Beziehung $\mu_4 = 3\mu_2^2$, und in obigen Formeln werden überall die ersten Glieder der rechten Seite verschwinden. Aus (13*a*) erhalten wir dann, zum Beispiel:

$$\sigma^2_{(\sigma^2,\,k)} = \frac{\mu_2^2\sqrt{2k\pi}}{N-k}.$$

Denkt man sich, also, k genügend und N beträchtlich groß und das Verteilungsgesetz "normal," so werden die mathematischen Erwartungen der empirischen

* Es ist also $S_{(1)}=1$; $S_{(2)}=10$; $S_{(3)}=138$; $S_{(4)}=1940$; $S_{(5)}=28130$; $S_{(6)}=414372$, u.s.w.

Streuungen der Größen $\sigma_k'^2$, σ'^2_{k+1}, σ'^2_{k+2}, ... sich etwa so zu einander verhalten, wie $\sqrt{k} : \sqrt{k+1} : \sqrt{k+2} : \ldots$.

In Tabelle I finden sich die bis zur 3^ten^ Dezimalstelle genauen Werte, welche Formel (12) für die ersten 6 endlichen Differenzen ergibt.

TABELLE I.

Ordnung der Differenz k	$\sigma^2_{(\sigma^2, k)}$ für ein beliebiges konstantes Verteilungsgesetz der zufälligen Variablen S	$\sigma^2_{(\sigma^2, k)}$ für ein "normales" Verteilungsgesetz der zufälligen Variablen S
$k=0$	$\frac{\mu_4 - \mu_2^2}{N}$	$\frac{2\mu_2^2}{N}$
$k=1$	$\frac{(N-1{,}500)\,\mu_4 + 0{,}500\,\mu_2^2}{(N-1)^2}$	$\frac{(3{,}000\,N - 4{,}000)\mu_2^2}{(N-1)^2}$
$k=2$	$\frac{(N-2{,}556)\,\mu_4 + (0{,}889\,N - 2{,}111)\,\mu_2^2}{(N-2)^2}$	$\frac{(3{,}889\,N - 9{,}778)\,\mu_2^2}{(N-2)^2}$
$k=3$	$\frac{(N-3{,}690)\,\mu_4 + (1{,}620\,N - 5{,}790)\,\mu_2^2}{(N-3)^2}$	$\frac{(4{,}620\,N - 16{,}860)\,\mu_2^2}{(N-3)^2}$
$k=4$	$\frac{(N-4{,}792)\,\mu_4 + (2{,}253\,N - 10{,}633)\,\mu_2^2}{(N-4)^2}$	$\frac{(5{,}253\,N - 25{,}012)\,\mu_2^2}{(N-4)^2}$
$k=5$	$\frac{(N-5{,}887)\,\mu_4 + (2{,}819\,N - 16{,}436)\,\mu_2^2}{(N-5)^2}$	$\frac{(5{,}819\,N - 34{,}094)\,\mu_2^2}{(N-5)^2}$
$k=6$	$\frac{(N-6{,}971)\,\mu_4 + (3{,}335\,N - 23{,}096)\,\mu_2^2}{(N-6)^2}$	$\frac{(6{,}335\,N - 44{,}008)\,\mu_2^2}{(N-6)^2}$

Aus Tabelle I erhalten wir die apriorischen *Streuungen* der Größen $\sigma_k'^2$. Um deren *mittlere Fehler* zu bestimmen, müssen aus den für erstere berechneten Zahlenwerten Quadratwurzeln gezogen werden. Die Größen μ_4 und μ_2 sind apriorisch und müssen beim Gebrauch der Tabelle in der Praxis durch empirische Annäherungswerte ersetzt werden. Dadurch entstehen Schwankungen, welche eigentlich ihrerseits mit Hilfe der mittleren Fehler der Streuungen $\sigma^2_{(\sigma^2, k)}$ gemessen werden sollten. Die Verfolgung dieses Weges würde uns aber hier zu weit führen.

(7) Formel (12) ergibt nur die apriorische Streuung (oder auch den mittleren Fehler) eines jeden $\sigma_k'^2$ gegenüber dem apriorischen μ_2. Sie besagt noch nichts über das gegenseitige Verhältnis der Größen $\sigma_k'^2$, σ'^2_{k+1}, σ'^2_{k+2}, ... welche, wie leicht ersichtlich, stark positiv miteinander korreliert sein müssen. Ihre mathematischen Erwartungen sind wohl alle einander gleich und diejenigen ihrer Differenzen, folglich, gleich Null.

In der Praxis werden jedoch die Differenzen dieser *empirischen* Größen von O mehr oder weniger beträchtlich abweichen. Um ein Maß dieses Abweichens zu erhalten, wollen wir die apriorischen Streuungen der Differenzen $\sigma'^2_{k+1} - \sigma_k'^2$ bestimmen:

$$E\,[(\sigma'^2_{k+1} - \sigma_k'^2) - O]^2 = E\,[(\sigma'^2_{k+1} - \mu_2) - (\sigma_k'^2 - \mu_2)^2]^2$$
$$= E\,(\sigma'^2_{k+1} - \mu_2)^2 + E\,(\sigma_k'^2 - \mu_2)^2 - 2E\,(\sigma_k'^2 - \mu_2)(\sigma'^2_{k+1} - \mu_2) \ldots\ldots (14).$$

Die beiden ersten Glieder der rechten Seite ergeben sich direkt aus Formel (12), für das dritte erhalten wir, wenn $k \leqq \frac{N}{2}$, folgenden Ausdruck:

$$E(\sigma_k'^2 - \mu_2)(\sigma'^2_{k+1} - \mu_2) = \frac{\mu_4 - 3\mu_2^2}{(N-k)}\left[1 - \frac{2S'_{(k)}}{C_{2k}^k C_{2k+2}^{k+1}(N-k-1)}\right] + \frac{2\mu_2^2}{N-k}\left[\frac{C_{4k+1}^{2k}}{C_{2k}^k C_{2k+2}^{k+1}} \cdot \frac{2N-2k-1}{N-k-1} - \frac{k+1}{2(N-k-1)}\right] \quad \ldots\ldots(15),$$

wobei

$$S'_{(k)} = \sum_{i=0}^{k-1}(C_k^i C_{k+1}^{i+2})^2 + 2\sum_{i=0}^{k-2}(C_k^i C_{k+1}^{i+3})^2 + 3\sum_{i=0}^{k-3}(C_k^i C_{k+1}^{i+4})^2 + \ldots + k(C_k^0 C_{k+1}^{k+1})^2,$$

und jedenfalls: $\frac{k+1}{2(N-k-1)} \geqq \frac{2S'_{(k)}}{C_{2k}^k C_{2k+2}^{k+1}(N-k-1)}$ * †.

Setzt man die entsprechenden Ausdrücke aus (12) und (15) in (14) ein, so erhält man, wenn man alle Größen von der Ordnung $\frac{1}{N^3}$ unterdrückt, für die ersten 6 Differenzen folgende Werte (s. Tabelle II).

TABELLE II‡.

Ordnung der Streuung	Formel für die apriorische Streuung, bis zur Ordnung $\frac{1}{N^2}$, für ein beliebiges konstantes Verteilungsgesetz der zufälligen Variablen S	Dieselbe, bis zur Ordnung $\frac{1}{N}$ abgerundet
$E(\sigma_1'^2 - \sigma_0'^2)^2 =$	$\frac{0{,}5000(\mu_4 - 3\mu_2^2)}{N(N-1)} + \frac{(N+1)\mu_2^2}{N(N-1)}$	$\frac{\mu_2^2}{N-1}$
$E(\sigma_2'^2 - \sigma_1'^2)^2 =$	$\frac{0{,}2778(\mu_4 - 3\mu_2^2)}{(N-1)(N-2)} + \frac{(0{,}2222N + 1{,}1111)\mu_2^2}{(N-1)(N-2)}$	$\frac{0{,}2222\mu_2^2}{N-2}$
$E(\sigma_3'^2 - \sigma_2'^2)^2 =$	$\frac{0{,}2544(\mu_4 - 3\mu_2^2)}{(N-2)(N-3)} + \frac{(0{,}1089N + 1{,}0933)\mu_2^2}{(N-2)(N-3)}$	$\frac{0{,}1089\mu_2^2}{N-3}$
$E(\sigma_4'^2 - \sigma_3'^2)^2 =$	$\frac{0{,}2096(\mu_4 - 3\mu_2^2)}{(N-3)(N-4)} + \frac{(0{,}0673N + 1{,}0808)\mu_2^2}{(N-3)(N-4)}$	$\frac{0{,}0673\mu_2^2}{N-4}$
$E(\sigma_5'^2 - \sigma_4'^2)^2 =$	$\frac{0{,}1873(\mu_4 - 3\mu_2^2)}{(N-4)(N-5)} + \frac{(0{,}0468N + 1{,}0721)\mu_2^2}{(N-4)(N-5)}$	$\frac{0{,}0468\mu_2^2}{N-5}$
$E(\sigma_6'^2 - \sigma_5'^2)^2 =$	$\frac{0{,}1693(\mu_4 - 3\mu_2^2)}{(N-5)(N-6)} + \frac{(0{,}0350N + 1{,}0656)\mu_2^2}{(N-5)(N-6)}$	$\frac{0{,}0350\mu_2^2}{N-6}$

Es ergibt sich also, daß, im Allgemeinen, je größer die Ordnung der endlichen Differenz k, desto geringer die mathematische Erwartung des Quadrates der

* Es ist also: $S_1' = 1$; $S_2' = 15$; $S_3' = 242$; $S_4' = 3815$; $S_5' = 59724$; etc.

† Formel (15) erlaubt es auch, den Korrelationskoeffizienten zwischen $\sigma_k'^2$ und σ'^2_{k+1} genau zu bestimmen.

‡ Für ein "normales" Verteilungsgesetz verschwinden alle Glieder, welche $\mu_4 - 3\mu_2^2$ enthalten, und die einzelnen Größen der Tabelle unterscheiden sich von denjenigen in Formel (23) meiner Monographie "Über ein neues Verfahren bei Anwendung der 'Variate-Difference'-Methode" (*Biometrika*, Vol. xv., 1923, p. 145) nur durch den Faktor $\frac{1}{(2k+1)^2}$. Die Formel (21) für $\sigma^2{}_0 D^2_{k+1}$ ist nämlich nicht

Differenz $\sigma'^2_{k+1} - \sigma_k'^2$ ausfällt. Das ist auch ganz natürlich, denn obgleich ihr möglicher minimaler Wert immer derselbe bleibt ($-\sigma_k'^2$ bei $\sigma'^2_{k+1} = 0$), ist ihr Maximum durch $+\frac{\sigma_k'^2}{2k+1}$ gegeben, verringert sich also mit wachsendem k*. Der mittlere Fehler der Differenz $\sigma'^2_{k+1} - \sigma_k'^2$ bleibt jedoch endlich, solange μ_2 und N endlich bleiben.

Ferner ist zu bemerken, daß die Differenz $\mu_4 - 3\mu_2^2$ sich hier bloß von der Ordnung $\frac{1}{N^2}$ erweist. Können bei beträchtlichem N auch alle Größen dieser Ordnung unterdrückt werden, so erhalten wir die Werte der letzten Kolonne der Tabelle II, welche, bis zur Größenordnug $\frac{1}{N}$, für ein beliebiges konstantes Verteilungsgesetz der Variablen gelten. Sie bestimmen sich, ganz allgemein, nach der Formel:

$$E(\sigma'^2_{k+1} - \sigma_k'^2)^2 = \frac{3k+1}{(2k+1)^3} \cdot \frac{C_{4k}^{2k}}{(C_{2k}^{k})^2} \cdot \frac{\mu_2^2}{N-k-1} \quad \ldots\ldots\ldots\ldots\ldots(16).$$

Ist k so groß, daß man die Stirling'sche Formel anwenden kann, so erhält man hieraus:

$$E(\sigma'^2_{k+1} - \sigma_k'^2)^2 = \frac{(3k+1)\,\mu_2^2\sqrt{2k\pi}}{2(2k+1)^3(N-k-1)} \quad \ldots\ldots\ldots\ldots\ldots(16a).$$

Schon bei $k = 6$ ergeben sich ganz annehmbare Resultate.

Die Formeln der Tabellen I und II vervollständigen nur einander, können sich aber gegenseitig nicht ersetzen. Will man feststellen, ob eine für die Theorie genügende Stabilität der Reihe $\sigma_k'^2$, σ'^2_{k+1}, σ'^2_{k+2}, ... erreicht ist, berechnet man die ersten Differenzen derselben und gebraucht die Formeln der Tabelle II. Ist es aber notwendig, die Grenzen der möglichen Abweichung des gefundenen *stabilen* $\sigma_k'^2$ von seiner mathematischen Erwartung—sagen wir, mit Hilfe des bekannten Tschebyscheff'schen Theorems—zu beurteilen, so wird man sich der Größen der Tabelle I bedienen müssen.

Ein anderes, in manchen Fällen interessantes, Charakteristikum derselben Reihe S bilden die Koeffizienten vom Typus:

$$R'_{j,k} = \frac{\frac{1}{(N-k-j)} \sum_{i=1}^{N-k-j} \Delta^k s_i \, \Delta^k s_{i+j}}{\frac{1}{(N-k)} \sum_{i=1}^{N-k} (\Delta^k s_i)^2} \quad \ldots\ldots\ldots\ldots\ldots(17),$$

wobei j eine beliebige ganze positive Zahl zwischen 0 und $(N-k-1)$ sein kann.

$E\{{}_0D_{k+1} - E\,{}_0D_{k+1}\}^2$ gleich, wie es dort fälschlich angenommen wurde, sondern der Größe:

$$\frac{(2k+1)^2\,E\{\sigma'^2_{k+1} - \sigma_k'^2\}^2}{\mu_2^2},$$

und zeigt, nach Bestimmung seiner Quadratwurzel, welch einen Teil der Größe μ_2 der mittlere Fehler der Differenz $(2k+1)\,\sigma'^2_{k+1} - (2k+1)\sigma_k'^2$ ausmachen kann. Nur in diesem Sinne darf sie angewandt werden. Vergl. darüber: O. Anderson, "Variate Difference Method," in den *Mélanges Pierre Struve, Recueil des Ecrits, présentés à M. Pierre Struve le* 30. *Janvier*, 1925, Prague, pp. 26, 27 (russisch).

* Durch die Formel dieses Maximums erklärt sich auch, weshalb, bei konkreten Messungen, die Reihe der Differenzen $\sigma_2'^2 - \sigma_1'^2$, $\sigma_3'^2 - \sigma_2'^2$, ... $\sigma'^2_{k+1} - \sigma_k'^2$ verhältnißmäßig oft auf dem Diagramm ein ganz hyperbelmäßiges Aussehen erhält.

Es kann nämlich bewiesen werden, daß $ER'_{j,k}$ desto näher zu

$$\frac{\frac{1}{(N-k-j)}\,E\sum_{i=1}^{N-k-j}\Delta^k s_i \Delta^k s_{i+j}}{\frac{1}{(N-k)}\,E\sum_{i=1}^{N-k}(\Delta^k s_i)^2}$$

herankommt, je größer N ist, um jedenfalls im Grenzfall, bei sehr großem N, mit diesem Ausdruck identisch zu werden*. Nun ist aber

$$\frac{1}{N-k-j}\,E\sum_{i=1}^{N-k-j}\Delta^k s_i \Delta^k s_{i+j} = (-1)^j\, C_{2k}^{k+j}\,\mu_2 \quad \text{und} \quad \frac{1}{N-k}\,E\sum_{i=1}^{N-k}(\Delta^k s_i)^2 = C_{2k}^{k}\,\mu_2 \qquad \text{.....(18).}$$

Daher erhalten wir für $ER'_{j,k}$ den Grenzwert:

$$ER'_{j,k} = (-1)^j \frac{C_{2k}^{k+j}}{C_{2k}^{k}} = (-1)^j \frac{k!\,k!}{(k-j)!\,(k+j)!}, \qquad \text{.....(19),}$$

* Der empirische Korrelationskoeffizient und die meisten ihm ähnlichen Gebilde besitzen die Eigenschaft, daß ihre Nenner und Zähler verschiedene Funktionen *derselben gegebenen Werte* $Z_1, Z_2, \ldots Z_N$ und $Z_1', Z_2', \ldots Z'_N$, sind. Die Wahrscheinlichkeit des Auftretens einer bestimmten i-ten Kombination von N zusammengehörigen Paaren Z_1 und Z_1', Z_2 und Z_2', $\ldots Z_N$ und Z'_N, eines, sagen wir, Korrelationskoeffizienten sei p_i. Und es seien nur n verschiedene Kombinationen dieser Paare, zu N in jeder, möglich, so daß $\sum_{i=1}^{n} p_i = 1$.

Bezeichnen wir einen Koeffizienten, entstanden aus der i-ten Kombination der Paare, durch $r_i = \frac{x_i}{y_i}$, so ist es ersichtlich, daß unter den gemachten Annahmen die Zähler x nur n verschiedene Werte: $x_1, x_2, \ldots x_n$ mit den entsprechenden Wahrscheinlichkeiten $p_1, p_2, \ldots p_n$ erhalten können; ist aber x_i gegeben so ist nur ein ganz bestimmtes einziges y_i möglich, dessen bedingte Wahrscheinlichkeit also gleich 1 zu setzen ist, so daß, folglich, auch die Zähler y ebenfalls nur n Werte: $y_1, y_2, \ldots y_N$, *mit denselben Wahrscheinlichkeiten* $p_1, p_2, \ldots p_n$ annehmen können. Dieselben Wahrscheinlichkeiten bestehen schließlich auch für die r_i selbst, so daß, also:

$$(1)\ Er = E\frac{x}{y} = p_1\frac{x_1}{y_1} + p_2\frac{x_2}{y_2} + \ldots + p_n\frac{x_n}{y_n}, \text{ und } (2)\ \frac{Ex}{Ey} = \frac{p_1x_1 + p_2x_2 + \ldots + p_nx_n}{p_1y_1 + p_2y_2 + \ldots + p_ny_n}.$$

Setzt man $\frac{x_i}{y_i} - Er = \epsilon_i$, oder $x_i = y_i Er + y_i\epsilon_i$, so verwandeln sich obige Ausdrücke in folgende:

$$(1)\quad E\frac{x}{y} = Er \cdot \sum_{i=1}^{n} p_i + \sum_{i=1}^{n} p_i\epsilon_i = E\frac{x}{y} + \sum_{i=1}^{n} p_i\epsilon_i \text{ (und folglich } \sum_{i=1}^{n} p_i\epsilon = 0),$$

und

$$(2)\quad \frac{Ex}{Ey} = \frac{Er \cdot \sum_{i=1}^{n} p_iy_i + \sum_{i=1}^{n} y_ip_i\epsilon_i}{Ey} = E\frac{x}{y} + \frac{\sum_{i=1}^{n} y_ip_i\epsilon_i}{Ey}.$$

Wir sehen also, daß, damit $E\frac{x}{y} = \frac{Ex}{Ey}$, notwendigerweise auch $\sum_{i=1}^{n} y_ip_i\epsilon_i$ gleich Null sein muß, wenn man, natürlich, vom Falle $Ey = 0$ absieht.

Wenn die Größen $y_1, y_2, \ldots y_n$ mit wachsendem N demselben Grenzwert (sagen wir, Ey) zustreben, so können hier die einzelnen y schließlich als einander gleich angesehen und vor das Summenzeichen genommen werden. Dann erhält man auch vermöge der Beziehung $\sum_{i=1}^{n} p_i\epsilon_i = 0$:

$$\sum_{i=1}^{n} y_ip_i\epsilon_i = Ey \cdot \sum_{i=1}^{n} p_i\epsilon_i = 0.$$

Mit wachsendem N kommen also dann die Ausdrücke $E\frac{x}{y}$ und $\frac{Ex}{Ey}$ sich immer näher, um im Grenzfalle einander gleich zu werden.

und folglich:

$$ER'_{1,k} = -\frac{k}{k+1},\quad ER'_{2,k} = +\frac{k(k-1)}{(k+1)(k+2)},\quad ER'_{3,k} = -\frac{k(k-1)(k-2)}{(k+1)(k+2)(k+3)} \text{ u.s.w.}^{*} \qquad \ldots\ldots\ldots\ldots(19a).$$

Es ist möglich, zu beweisen†, daß wenn $ER'_{1,k}$ den eben angegebenen Wert besitzt, auch unbedingt $E\sigma'^{2}_{k+1} = E\sigma_k'^{2}$; wenn, ferner, $ER'_{2,k}$, $ER'_{3,k}$, ... $ER'_{j,k}$ die in Form. (19) angegebenen Werte annehmen, so ist auch, ganz allgemein,

$$E\sigma'^{2}_{k+2} = E\sigma'^{2}_{k+3} = \ldots = E\sigma'^{2}_{k+j}.$$

Folglich, können die Koeffizienten $R'_{i,k}$, wenn ein entsprechendes Kriteriensystem aufgebaut ist, die schwierige Berechnung der höheren endlichen Differenzen in einigen Fällen ersetzen und dadurch die notwendige Rechenarbeit erleichtern‡.

(8) Außer der statistischen Reihe $u_1, u_2, \ldots u_N$, sei noch eine andere Reihe: $v_1, v_2, \ldots v_N$, gegeben. Sie bestehe ebenfalls aus einer "glatten" Komponente G' und einer "restlichen" s', so daß $v_i = G_i' + s_i'$. Es sei festgestellt, daß in der k-ten endlichen Differenz keine Reste von G' mehr verblieben seien, so daß $\Delta^k v_i = \Delta^k s_i'$. Außerdem sei angenommen, daß auch die Reihe s' als eine solche Reihe angesehen werden könne, welche eine zufällige Variable mit einem konstanten Verteilungsgesetze bei N von einander unabhängigen Versuchen ergibt.

Führt man, in Anlehnung an Prof. Tchouproff §, die Bezeichnungen ein:

$$\left.\begin{aligned} m_{g/h} &= Es_i^{g}\, s_i'^{h} \text{ (und folglich: } m_{g/0} = Es^{g},\ m_{0/h} = Es'^{h}) \\ m^{(j)}_{g/h} &= Es_i^{g}\, s'^{h}_{i+j} \\ \mu_{g/h} &= E(s_i - m_{1/0})^{g}(s_i' - m_{0/1})^{h} \text{ (und folglich:} \\ &\qquad \mu_{g/0} = E(s - m_{1/0})^{g},\ \mu_{0/h} = E(s' - m_{0/1})^{h}) \\ \mu^{(j)}_{g/h} &= E(s_i - m_{1/0})^{g}(s'_{i+j} - m_{0/1})^{h} \end{aligned}\right\} \ldots\ldots(20),$$

* Um $\frac{1}{N-k-j} E \sum_{i=1}^{N-k-j} \Delta^k s_i \Delta^k s_{i+j}$ zu bestimmen, ist es hauptsächlich notwendig, den Satz: $\sum_{i=0}^{k-j} C_k^i C_k^{i+j} = C_{2k}^{k+j}$ zu beweisen. Dieser Satz ist nicht neu (es dürfte überhaupt nicht leicht sein, in der Theorie der Kombinatorik etwas neues zu sagen) und findet sich, zum Beispiel, in Netto's *Lehrbuch der Kombinatorik*. Am leichtesten ist er aus der Betrachtung der Identität $(1+t)^k \left(1+\frac{1}{t}\right)^k = \frac{(1+t)^{2k}}{t^k}$ zu beweisen. Entwickelt man nämlich die linke Seite, so ist: $\sum_{i=0}^{k-j} C_k^i C_k^{i+j}$ der Koeffizient vor t^j; entwickelt man die rechte Seite, so steht vor t^j der Koeffizient: C_{2k}^{k+j}.

† Vergl. O. Anderson, *Über ein neues Verfahren, etc.*, S. 138—140.

‡ Näheres darüber siehe in der eben zitierten Schrift. Die dort benutzten Kriterien $\frac{{}_0D_i}{2i-1}$ und $\frac{{}_kD_{i-k}}{2i-1}$ bestimmen sich aus $\frac{\sigma_i'^{2} - \sigma'^{2}_{i-1}}{\sigma'^{2}_{i-1}}$.

§ Al. A. Tchouproff, "Aufgaben und Voraussetzungen der Korrelationsrechnung" (*Nordisk Statistisk Tidskrift*, 1923, Bd. 2, Häft 1), S. 52, Anhang.

ferner die Bezeichnungen:

$$\left.\begin{aligned} p'_k &= \frac{1}{C^k_{2k}} \cdot \frac{\sum_{i=1}^{N-k} \Delta^k s_i \, \Delta^k s_i'}{N-k} \\ \text{und} \quad p'_{j\,(k)} &= \frac{1}{C^k_{2k}} \cdot \frac{\sum_{i=1}^{N-k-j} \Delta^k s_i \, \Delta^k s'_{i+j}}{N-k-j} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(21),$$

so kann es leicht bewiesen werden, daß:

$$Ep_k' = \mu_{1/1},$$

wenn nur s_i und s'_{i+j}, bei jedem $j \neq 0$, vollkommen unabhängig von einander sind und folglich $\mu^{(j)}_{1/1} = 0$ bei $j \neq 0$.

Unter denselben Bedingungen wird auch:

$$\sigma^2_{(p,\,k)} = E(p_k' - \mu_{1/1})^2 = \frac{\mu_{2/2} - \mu_{2/0}\,\mu_{0/2} - 2\mu^2_{1/1}}{N-k}\left[1 - \frac{2S_k}{(C^k_{2k})^2 (N-k)}\right]$$
$$+ \frac{(\mu^2_{1/1} + \mu_{2/0}\,\mu_{0/2})}{N-k}\left[\frac{C^{2k}_{4k}}{(C^k_{2k})^2} - \frac{k}{2(N-k)}\right] \quad \ldots\ldots\ldots\ldots(22).$$

Der Aufbau dieser Formel ist genau derselbe, wie bei (12). Nur muß man $\frac{\mu^2_{1/1} + \mu_{2/0}\,\mu_{0/2}}{2}$, statt μ_2^2, und $(\mu_{2/2} - \mu_{2/0}\,\mu_{0/2} - 2\mu^2_{1/1})$, statt $(\mu_4 - 3\mu_2^2)$ setzen. Letzterer Ausdruck verschwindet ebenfalls, wie $(\mu_4 - 3\mu_2^2)$, im Falle einer "normalen" Korrelationsfläche, wo die bekannte Relation $r_{2/2} = 1 + 2r^2_{1/1}$ besteht*. Mit den soeben erwähnten Substitutionen können also auch hier die Formeln der Tabelle I benutzt werden.

Mit denselben Substitutionen sind ebenfalls die Formeln der Tabelle II auf die mathem. Erwartungen der Quadrate der Differenzen: $(p_k' - p'_{k-1})$ anwendbar.

Desgleichen erhalten wir ferner:

$$Ep_k' = Ep'_{k+1} = Ep'_{k+2} = \ldots = Ep'_{k+j},$$

wenn:

$$\frac{\mu^{(-1)}_{1/1} + \mu^{(+1)}_{1/1}}{2\mu_{1/1}} = -\frac{k}{k+1}, \quad \frac{\mu^{(-2)}_{1/1} + \mu^{(+2)}_{1/1}}{2\mu_{1/1}} = \frac{k(k-1)}{(k+1)(k+2)}, \ldots \frac{\mu^{(-j)}_{1/1} + \mu^{(+j)}_{1/1}}{2\mu_{1/1}}$$
$$= (-1)^j \frac{k!\,k!}{(k-j)!\,(k+j)!}\dagger \quad \ldots\ldots\ldots\ldots(23).$$

* S. unten: Formel (24). Bei Anwendung letzterer Formel kann man (22) auch folgendermaßen darstellen:

$$\sigma^2_{(p,\,k)} = \frac{\mu_{2/0}\,\mu_{0/2}\,(r_{2/2} - 1 - 2r^2_{1/1})}{N-k}\left[1 - \frac{2S_k}{(C^k_{2k})^2 (N-k)}\right] + \frac{\mu_{2/0}\,\mu_{0/2}\,(1 + r^2_{1/1})}{N-k}\left[\frac{C^{2k}_{4k}}{(C^k_{2k})^2} - \frac{k}{2(N-k)}\right] \ldots(22a).$$

† O. Anderson, *Über ein neues Verfahren, etc.*, S. 138.

Definiert man:

$$r_{g/h} = \frac{\mu_{g/h}}{[\mu_{2/0}]^{\frac{1}{2}g} \cdot [\mu_{0/2}]^{\frac{1}{2}h}} \left(\text{und folglich: } r_{1/1} = \frac{\mu_{1/1}}{\sqrt{\mu_{2/0}\,\mu_{0/2}}}\right) \quad \text{.....(24),}$$

$$r^{(j)}_{g/h} = \frac{\mu^{(j)}_{g/h}}{[\mu_{2/0}]^{\frac{1}{2}g} \cdot [\mu_{0/2}]^{\frac{1}{2}h}}; \quad {}_0\rho_j = \frac{\frac{1}{N-j}\sum_{i=1}^{N-j}(s_i - m_{1/0})(s'_{i+j} - m_{0/1})}{\sqrt{\frac{1}{N}\sum_{i=1}^{N}(s_i - m_{1/0})^2 \cdot \frac{1}{N}\sum_{i=1}^{N}(s_i' - m_{0/1})^2}}$$

und

$$\rho_j = \frac{\frac{1}{N-j}\sum_{i=1}^{N-j}(s_i - s_{(N)})(s'_{i+j} - s'_{(N)})}{\sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(s_i - s_{(N)})^2 \cdot \frac{1}{N-1}\sum_{i=1}^{N}(s_i' - s'_{(N)})^2}} \quad \text{.....(25),}$$

wenn $s_{(N)} = \frac{\sum_{i=1}^{N} s_i}{N}$*, und ferner:

$$\rho_{j\,(k)} = \frac{p'_{j\,(k)}}{\sqrt{{}_1\sigma_k'^2 \cdot {}_2\sigma_k'^2}} \quad \text{.....(26)}$$

(wobei ${}_1\sigma_k'^2$ bei Anwendung der Formel (11) auf die Reihe $s_1, s_2, \ldots$, und ${}_2\sigma_k'^2$ bei Anwendung derselben Formel auf die Reihe $s_1', s_2', \ldots$ entsteht),—so kann man leicht beweisen, daß

$$E\rho_{0\,(k)} = E\frac{p_k'}{\sqrt{{}_1\sigma_k'^2 \cdot {}_2\sigma_k'^2}} = r_{1/1},$$

wenn man nur davon absieht, daß $E\rho_{0\,(k)}$ bloß in erster Annäherung dem Ausdruck $\frac{Ep_k'}{\sqrt{E_1\sigma_k'^2 \cdot E_2\sigma_k'^2}}$ gleich gesetzt werden kann†.

Was die Größe $\sigma^2_{(r,k)} = E(\rho_{0\,(k)} - r_{1/1})^2$ anbetrifft, so ergibt sie sich—in erster Annäherung, für eine "normale" Verteilung und mit allen Reserven, die durch den neuesten Stand der Forschung geboten sind,—aus der Formel:

$$\sigma^2_{(r,k)} = \frac{(1 - r^2_{1/1})^2}{N-k}\left[\frac{C^{2k}_{4k}}{(C^k_{2k})^2} - \frac{k}{2(N-k)}\right] \quad \text{.....(27).}$$

Auch hier können, also, die Formeln der letzten Kolonne der Tabelle I benutzt werden, wenn man dort μ_2^2 durch $\frac{(1 - r^2_{1/1})^2}{2}$ ersetzt‡.

* Wenn j negativ ist, so muß, statt $\frac{1}{N-j}$, überall $\frac{1}{N+j}$ genommen werden.

† Vergleich hierüber besonders die neuesten (russischen) Arbeiten Prof. A. Tchouproff's: *Über die mathematische Erwartung des Quotienten zweier gegenseitig nicht unabhängigen zufälligen Variablen*, und *Das Ausgangsproblem der mathematischen Theorie der verschiedenen Verfahren für die statistische Untersuchung der Zusammenhänge zwischen zwei zufälligen Variablen*. Ferner E. Slutsky, *Über einige Korrelationsschemen und über den systematischen Fehler des empirischen Korrelationskoeffizienten*.

‡ Werden demzufolge die vor N stehenden Koeffizienten dieser Kolonne durch 2 dividiert, so erhält man ganz richtig die von A. Ritchie-Scott im XI. Bande der *Biometrika* ("Note on the probable error of the Coefficient of Correlation in the Var. Diff. Corr. Method") berechneten Funktionen: $2\phi(m) - 1$.

Ist schließlich N und k genügend groß, so kommt man auch zur Formel:

$$\sigma^2_{(r,k)} = \frac{(1-r^2_{1/1})^2}{N-k}\sqrt{\frac{k\pi}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(27a).$$

Die Annahme der gegenseitigen vollkommenen Unabhängigkeit aller s_i und s_j', ausgenommen den Fall $i=j$, kann aber lange nicht immer gerechtfertigt werden. Es ist sehr wohl möglich, daß, obgleich jede der beiden zufälligen Variablen ein konstantes Verteilungsgesetz aufweist und bei jeder die einzelnen Versuche von einander vollkommen unabhängig sind, es dennoch zwischen verschiedenen s_i und s'_{i+j} eine gewisse Korrelation bestehen könne. So vermag man, zum Beispiel, aus der Reihe s_1, s_2, s_3, folgende zwei Reihen zu bilden:

$$\begin{aligned} x_1 &= s_1 + s_2 + s_3\,, & y_1 &= s_1 + s_5 + s_9\,,\\ x_2 &= s_4 + s_5 + s_6\,, & y_2 &= s_4 + s_8 + s_{12},\\ x_3 &= s_7 + s_8 + s_9\,, \text{ und} & y_3 &= s_7 + s_{11} + s_{15},\\ x_4 &= s_{10} + s_{11} + s_{12}, & y_4 &= s_{10} + s_{14} + s_{18},\\ &\ldots\ldots\ldots\ldots & &\ldots\ldots\ldots\ldots \end{aligned}$$

Wenn die s von einander unabhängig sind, so sind auch alle x von einander unabhängig, und ebenso alle y. Und trotzdem hat jedes y_i je eine gemeinsame Komponente nicht nur mit jedem x_i, sondern auch mit x_{i+1} und x_{i+2}.

In solchen Fällen (und sie brauchen nicht selten vorzukommen) wird also weder die Reihe der Koeffizienten $E\rho_{0\,(k)}$, $E\rho_{0\,(k+1)}$, ... konstant, noch wird einer derselben $r_{1/1}$ ergeben.

Um hier die Differenzenmethode anzuwenden, muß man entweder zu E. S. Pearson's Funktionen $\phi(n, \rho_0, \rho)$* oder, was auf dasselbe hinausläuft, zu den von mir im Artikel "Über ein neues Verfahren," etc.†, entwickelten Formeln greifen. Der Gebrauch letzterer wird weiter unten an einigen konkreten Beispielen illustriert. Hier seien sie nur des Zusammenhanges wegen angeführt.

Bei größeren N und unter den Voraussetzungen des § 10 (siehe dort) kann mit ganz guter Annäherung gesetzt werden:

$$p'_{j\,(k)} = \frac{(-1)^k}{C^k_{2k}} \cdot {}_1\sigma' {}_2\sigma' \{C^{2k}_{2k}\rho_{k+j} - C^{2k-1}_{2k}\rho_{k+j-1} + C^{2k-2}_{2k}\rho_{k+j-2} - \ldots + C^0_{2k}\rho_{j-k}\} \ldots(28),$$

$$\sigma_k'^2 = \frac{1}{C^k_{2k}} \cdot \sigma'^2 \cdot \{C^k_{2k} - 2C^{k+1}_{2k}R_1' + 2C^{k+2}_{2k}R_2' - \ldots \pm 2C^{2k}_{2k}R_k'\} \quad \ldots\ldots\ldots\ldots(29),$$

wenn
$$R_j' = \frac{\frac{1}{N-j}\sum_{i=1}^{N-j}(s_i - s_{(N)})(s_{i+j} - s_{(N)})}{\frac{1}{N-1}\sum_{i=1}^{N}(s_i - s_{(N)})^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(29a),$$

und ${}_1\sigma'$ und ${}_2\sigma'$ die auf die Reihen s und s' angewandte Formel (10) bedeuten.

* K. Pearson and E. M. Elderton, "On the Variate-Difference Method" (*Biometrika*, Vol. XIV. March 1923, p. 294), Form. (xviii) und ff.

† *Loc. cit.*, pp. 136, 137.

(9) Gehen wir nun weiter. Angenommen, die empirischen Formeln für

$$\sigma_k'^2,\ \sigma'^2_{k+1},\ \ldots,\quad \sigma'^2_{k+1}-\sigma_k'^2,\ \sigma'^2_{k+2}-\sigma'^2_{k+1},\ \ldots,\quad R'_{1,k},\ R'_{2,k},\ \ldots,$$
$$p_k',\ p'_{k+1},\ \ldots,\ \rho_{0(k)},\ \rho_{0(k+1)},\ \ldots,\ \text{etc.}$$

entsprechen "m. B.," das heißt: *modo Bernoulliano**, dem Schema einer zufälligen Variablen mit einem konstanten Verteilungsgesetz und N von einander unabhängigen Versuchen. Ist dadurch schon bewiesen, daß die ursprüngliche Reihe $u_1, u_2, \ldots u_N$ eine solche Komponente wirklich besitzt? Ist es überhaupt sicher, daß die gefundenen Werte, wenn auch ebenfalls nur "modo Bernoulliano," die gesuchten sind? Und wenn nicht, so was haben wir denn eigentlich erreicht?

Die erste und zweite Frage sind entschieden zu verneinen. Es wäre naiv zu glauben, daß die Differenzenmethode *oder eine beliebige andere Methode* imstande wären, das Kunststück zu vollbringen,—ohne irgendwelche apriorische Kenntnisse über die zu zerlegende Reihe zu besitzen (über solche Kenntnisse verfügt man in der Praxis nur in den seltensten Fällen) und ohne mehr oder weniger gewagte Hypothesen aufzustellen,—aus N gegebenen Gleichungen vom Typus $u_i = G_i + s_i$ ganze $2N$ Unbekannte ("G" und "s") zu bestimmen†. Ohne derartige Hypothesen kann man eben beim Zerlegen statistischer Reihen nicht auskommen. Aber gerade in dieser Beziehung ermöglicht die Anwendung der Differenzenmethode, einen Schritt weiter zu tun, als es sonst in der Regel üblich ist, denn bei ihr wird das evident gehalten, was bei anderen Methoden nur im stillen und unbewußt hineingeschmuggelt wird.

(1) Eine jede gegebene statistische Reihe von N Gliedern kann durch eine Parabel, nicht höher als $(N-1)$ Ordnung genau wiedergegeben werden. (2) Eine jede gegebene statistische Reihe kann durch eine Fourier-Serie oder durch eine andere trigonometrische Funktion genau dargestellt werden. Je höher die Ordnung der Parabel, je größer die Zahl der Parameter in einer trigonometrischen Funktion, desto genauer wird sich, im Allgemeinen, die berechnete "ausgeglichene" Kurve den Gliedern einer gegebenen Reihe anpassen und desto geringere Werte werden die *scheinbaren* "restlichen Komponenten" aufweisen, bis sie, endlich, ganz verschwinden. Wo hat hier der Forscher Halt zu machen? Wo, bei welcher Potenz der Parabel, bei welcher Parameter-Zahl der trigonometrischen Formel hat er Grund festzustellen: dies hier ist ungefähr die "glatte" Komponente und das dort dürfte nur eine Funktion der wahren "restlichen Komponente" ausmachen? So naheliegend eigentlich diese Frage ist, kann man doch nicht behaupten, daß

* Ein Ausdruck Prof. W. Romanowsky's, der in der russischen statistischen Litteratur sich einzubürgern scheint.

† Wie es die neueren Untersuchungen Prof. Al. A. Tchouproff's und eines seiner Schüler (J. Morduch) erwiesen haben, kann man, überhaupt—sogar aus dem Studium einer einfachen (nicht zusammengesetzten) statistischen Reihe heraus—lange nicht alle Eigenschaften der ihr zu Grunde liegenden zufälligen Variablen absolut sicher bestimmen. So, zum Beispiel, ist es direkt unmöglich, den Fall der gegenseitigen Unabhängigkeit der an einer zufälligen Variablen angestellten Versuche von einigen anderen Fällen, etwa vom Falle einer "uniformen Verbundenheit" der Versuche, zu unterscheiden. (Vergl. Al. A. Tchouproff, "Ist die normale Stabilität empirisch nachweisbar?" in *Nordisk Statistisk Tidskrift*, Band 1, 1922, Häf. 3, 4.) Es liegt hier sogar kein bescheidenes "ignoramus," sondern geradezu ein anmaßendes "ignorabimus" vor.

sie bis jetzt eine eindeutige Antwort erhalten hat. Viele Forscher machen eben gerade dort halt, wo es in ihre Theorien am besten paßt, oder wo ihnen die Rechenarbeit beschwerlich zu werden beginnt. Und da die menschliche Geduld eine stark veränderliche Größe ist, so bleibt der eine schon bei einer Geraden stehen und geht der andere bis in die hohen Parabelordnungen hinein.

Die Differenzenmethode erlaubt hier eine objektivere Stellung einzunehmen. Sich ihrer bedienend, kann der Forscher etwa folgendes feststellen. Von einer gewissen k-ten Differenz angefangen, sind bestimmte Koeffizienten für eine oder für mehrere statistische Reihen ungefähr stabil geworden. Das *könnte* bedeuten, daß die "glatte" Komponente angenähert durch eine Parabel $(k-1)$-ter Ordnung darstellbar sei, und das *könnte* ferner bedeuten, daß, die "restliche" Komponente als eine zufällige Variable mit konstantem Fehlergesetz und gegenseitiger Unabhängigkeit der einzelnen Versuche angesehen werden könne. Und je länger die Reihe der stabilen Koeffizienten ("modo Bernoulliano," natürlich),—desto wahrscheinlicher der Schluß. Oder: die Reihe der Koeffizienten wird nicht stabil; es ist also *unwahrscheinlich*, daß die "restliche" Komponente eine derartige zufällige Variable sei; wahrscheinlicher sei, daß sie diese oder jene anderen Eigenschaften besitze. Oder: die mittleren Fehler sind zu groß und das Material läßt verschiedene Deutungen zu; es ist, folglich, für eine Zerlegung in einzelne Komponenten überhaupt wenig geeignet, u.s.w.*

Haben solche Feststellungen theoretischen Wert? In manchen Fällen doch wohl! Können sie dadurch widerlegt werden, daß man etwa beweist, daß kurzperiodische Sinus-Komponenten unerklärbaren Ursprunges die zufällige Komponente in gewissen Fällen darstellen oder verdecken können? Ich glaube nicht. Denn, erstens, ist die Annahme der Existenz einer zufälligen Beobachtungskomponente mit gegenseitiger Unabhängigkeit der Glieder doch schließlich diejenige Hypothese, auf welcher bis jetzt, trotz aller Kritik, der größte Teil der Fehlertheorie aufgebaut ist; und, zweitens, ist die vorgeschlagene Hypothese jedenfalls nicht besser begründet. Kein Forscher verwirft aber eine alte Hypothese, die einen gegebenen Erscheinungskomplex befriedigend erklärt, nur deshalb, weil derselbe auf eine andere, nicht weniger künstliche, Weise *auch* erklärt werden könnte†.

* Die Frage, ob das gegebene Material, bei Annahme einer bestimmten Formel für die "glatte" Komponente, noch die Hypothese der Anwesenheit einer "zufälligen" restlichen Komponente überhaupt zuläßt, wurde bis jetzt meistenteils doch wohl nur von denjenigen Forschern untersucht, welche bei Sterblichkeitstafel-Ausgleichungen das Lexis'sche Kriterium Q^2 benutzten. So, zum Beispiel, in Rußland von Herrn B. Jastremsky (seine Methode zur "Auffindung des sich verändernden Niveau's einer statistischen Reihe" läuft letzten Endes auch auf die Berechnung von Q^2 hinaus). Es fragt sich nur ob der Lexis'sche Divergenzkoeffizient für einen solchen Gebrauch wirklich geeignet ist (vergl. unten § 12).

† Und vollkommends geht es nicht an, nur auf Grund der theoretischen *Möglichkeit* eines störenden Einflusses von, sagen wir, Yules kurzperiodischen Sinus-Komponenten, behaupten zu wollen, daß diese auch wirklich stören. Ihre Existenz müßte zu allererst nachgewiesen werden, was bis jetzt einwandfrei noch nicht gelungen ist. Ein drastischer Vergleich. Wenn ein Meteorstein auf eine Lokomotive herabfällt, so dürfte es voraussichtlich ein Eisenbahnunglück geben. Folgt aber daraus, daß man in der Eisenbahn nicht fahren soll, *weil* Meteorsteine auf Lokomotiven fallen?

(10) Zur Kontrolle, untersuchen wir jetzt die Frage, was man *mit Sicherheit* aus einem bestimmten Verhalten der Größen $\sigma_k'^2$, σ'^2_{k+1} u.s.w. über die Beschaffenheit der restlichen Komponente folgern kann. Wie ein jedes "umgekehrtes" Problem, ist diese Frage schwieriger zu behandeln, als die "direkte."

Das denkbar allgemeinste stochastische Schema zur Darstellung der restlichen Komponente dürfte wie folgt lauten: "An N zufälligen Variablen—$S_1, S_2, \ldots S_N$—werden N Versuche vorgenommen—je ein Versuch an jeder derselben. Die Variablen mögen beliebigen Verteilungsgesetzen folgen und unter einander in beliebigen Abhängigkeitsbeziehungen stehen*." Wie es aber Professor A. Tchouproff selbst gezeigt hat†, kommt aus einem derartigen Schema nichts rechtes heraus, da, ohne bestimmte Annahmen über die Verteilungsgesetze und die gegenseitige Verbundenheit der Versuche, die Formeln immer beträchtlich mehr Unbekannte aufweisen werden, als es verschiedene Gleichungen aufgestellt werden können. Führen wir deshalb noch 2 beschränkende Bedingungen ein. Erstens, mögen die mathematischen Erwartungen beliebiger Potenzen aller Variablen konstant sein:

$$Es_1^h = Es_2^h = \ldots Es_N^h, \text{ bei } h = 1, 2, 3, \ldots$$

und zweitens, möge die Differenz

$$Es_i s_{i+j} - Es_i\, Es_{i+j} = E(s_i - Es)(s_{i+j} - Es),$$

von i unabhängig und bloß von j abhängig sein‡.

* Vergl. Al. A. Tchouproff, "Zur Theorie der Stabilität statistischer Reihen," *Skandinavisk Aktuarietidskrift*, 1919, S. 203.

† Al. A. Tchouproff, "On the mathematical expectation of the moments of frequency distributions in the case of correlated observations," *Metron*, Vol. II. N. 3—4, 1923, pp. 7, 8.

‡ Vergl. Tchouproff, "Zur Theorie der Stabilität, etc.," S. 209. O. Anderson, "Über ein neues Verfahren, etc.," S. 137. (Die beschränkenden Bedingungen sind in der letzten Schrift etwas allgemeiner gehalten, als oben. Der Gewinn ist aber nicht besonders groß und verdunkelt nur die Formeln.) Czuber regt ähnliches an. Auch bei A. A. Markoff finden sich ähnliche Gedankengänge. Dadurch, daß nicht alle $Es_i s_{i+j} - Es_i Es_{i+j}$ bei beliebigen j einander gleich gesetzt werden (was zu einer "uniform-verbundenen Reihe" führt), sondern daß "Ketten" gebildet werden:

$$Es_1 s_2 - Es_1 Es_2 = Es_2 s_3 - Es_2 Es_3 = Es_3 s_4 - Es_3 Es_4 = \ldots = Es_{N-1} s_N - Es_{N-1} Es_N,$$
$$Es_1 s_3 - Es_1 Es_3 = Es_2 s_4 - Es_2 Es_4 = Es_3 s_5 - Es_3 Es_5 = \ldots = Es_{N-2} s_N - Es_{N-2} Es_N,$$
$$\ldots\ldots\ldots

Bezeichnet man noch

$$r_j = \frac{E(s_i - Es)(s_{i+j} - Es)}{E(s_i - Es)^2}, \text{ (folglich, ist } r_0 = 1) \quad \ldots\ldots\ldots\ldots (30),$$

und denkt sich diese Koeffizienten in der Reihenfolge

$$r_{k+j},\ r_{k+j-1},\ r_{k+j-2},\ \ldots\ r_j,\ r_{j-1},\ r_{j-2},\ \ldots \quad \ldots\ldots\ldots\ldots (31),$$

geordnet, so erhält man leicht die beiden apriorischen Formeln, welche bei den gegebenen Annahmen genau sind, den empirischen Annäherungs-Formeln (28) und (29) zu Grunde liegen und, schließlich, diejenigen Bedingungsgleichungen ergeben, welchen die Koeffizienten r_j genügen müssen, damit genau

$$E\sigma_i'^2 = E\sigma'^2_{i+1} = \ldots :$$

$$\frac{E\sum_{i=1}^{N-k-j} \Delta^k s_i \Delta^k s_{i+j}}{N-k-j} = \mu_2 (-1)^k \Delta^{2k} r_{k+j} \text{ und } \frac{E\sum_{i=1}^{N-k} \Delta^k s_i^2}{N-k} = \mu_2 (-1)^k \Delta^{2k} r_k \ldots^* \quad \ldots\ldots\ldots\ldots (32).$$

Die genannten Bedingungsgleichungen können in zwei verschiedenen Systemen dargestellt werden, die auf einander unschwer zurückzuführen sind:

$$\left.\begin{aligned}
&\text{bei } i=1,\ -r_1 + r_2 = 0\\
&\text{„ } i=2,\ -5r_1 + 8r_2 - 3r_3 = 0\\
&\text{„ } i=3,\ -7r_1 + 14r_2 - 9r_3 + 2r_4 = 0\\
&\text{„ } i=4,\ -42r_1 + 96r_2 - 81r_3 + 32r_4 - 5r_5 = 0\\
&\quad \ldots\ldots\ldots\

wird; es ist also $\nabla' r_1$ gleich $r_1 - r_2$ und nicht $r_1 - r_0$, $\nabla'' r_1$ gleich $r_1 - 2r_2 + r_3$ und nicht $r_1 - 2r_0 + r_{-1}$, $\nabla''' r_1$ gleich $r_1 - 3r_2 + 3r_3 - r_4$ und nicht $r_1 - 3r_0 + 3r_{-1} - r_{-2}$, u.s.w. Diese Bezeichnung ist mit der von den Aktuarien gewöhnlich gebrauchten nicht zu verwechseln. Es ist noch zu beachten, daß die unterste Formel in (33) dem Falle $i = k - 1$ und in (33a) dem Falle $i = k$ entspricht.

Ist schon $E\sigma_1'^2 = E\sigma_2'^2 = E\sigma_3'^2 = \ldots = E\sigma_i'^2$, so entspricht das dem Falle $i = 1$, und alle Bedingungsgleichungen (33) und (33a) müssen erfüllt sein. Ist erst, zum Beispiel, $E\sigma_3'^2 = E\sigma_4'^2 = \ldots = E\sigma_i'^2$, so entspricht das dem Falle $i = 3$ und alle Gleichungen des Systems (33) müssen erfüllt sein, *ausgenommen die beiden ersten*, wo die linken Seiten nicht Null zu sein brauchen. Ebenso im System (33a). Andererseits, sind die Bedingungsgleichungen von einem gewissen i abwärts erfüllt, so müssen auch die Beziehungen bestehen: $\sigma_i'^2 = \sigma'^2_{i+1} = \sigma'^2_{i+2} = \ldots$. Reihen, für welche solche Beziehungen gelten, bezeichne ich als zur "Gruppe R" gehörig.

Der Inhalt der Bedingungsgleichungen (33) und (33a) ist nicht so leicht zu erschöpfen, und wir werden uns hier nur mit dem Nötigsten befassen.

Ist i gleich 1, so verfügen wir über $k-1$ verschiedene Gleichungen zur Bestimmung von k Unbekannten: $r_1, r_2, \ldots r_k$. Wir erhalten aus ihnen leicht:

$$r_1 = r_2 = r_3 = \ldots = r_k,$$

das heißt, die Bedingungen einer "uniformen Reihe" Tchouproff's.

Ist i gleich 2, so besitzen wir nur $(k-2)$ Gleichungen für k Unbekannte und kommen zum System:

$$r_3 - r_4 = \frac{7}{3}(r_1 - r_2);\ r_4 - r_5 = \frac{9}{3}(r_1 - r_2);$$

$$r_5 - r_6 = \frac{11}{3}(r_1 - r_2);\ \ldots\ r_{k-1} - r_k = \frac{(2k-1)}{3}(r_1 - r_2).$$

Die Reihe $r_1, r_2, r_3, \ldots r_k$ wird, folglich, durch gewisse Parabeln 2-ter Ordnung genau ausgedrückt. Wie man sieht, werden dabei die Differenzen $r_i - r_{i+1}$, welche durchwegs dasselbe Vorzeichen behalten, mit wachsendem i immer größer. Ein derartiges Verhalten der Reihe r entspricht so wenig demjenigen, welches wir von der Wirklichkeit erwarten können, daß es wohl bei weitem wahrscheinlicher bleibt, daß $r_1 - r_2 = 0$. Wir kehren, folglich, auch hier zum System der "uniformen" Verbundenheit der Versuche zurück.

Ist, überhaupt, i gleich j, so verfügt man über nicht mehr als $k-j$ Gleichungen zur Bestimmung von k Unbekannten. Es können, folglich, nur je $(k-j)$ Unbekannte durch gewisse lineare Funktionen der übrigen j ausgedrückt werden. Es ist aber möglich, ganz allgemein zu beweisen, daß auch in diesem Falle die Reihe (31) durch Parabeln jedenfalls nicht höher, als $(2j-1)$-ter Ordnung darstellbar sein müßte*. Je größer die letzte erreichte Differenz k im Vergleich zu $(2j-1)$ ist, desto weniger wahrscheinlich wird ein so gesetzmäßiges Verhalten

* Ebenda, S. 146 Anmerkung.

der Reihe der r, und desto wahrscheinlicher, also, die frühere Annahme einer "uniformen" Reihe*.

Wenden wir uns jetzt den Bedingungen zu, unter welchen die Reihe $E\sigma_1'^2, E\sigma_2'^2, E\sigma_3'^2, \ldots$ *nicht konstant werden kann.*

Es kann bewiesen werden†, daß wenn, von einem gewissen i angefangen, in System (33) und (33a) die linken Seiten aller Gleichungen > 0 werden, auch $E\sigma_i'^2 < E\sigma'^2_{i+1} < E\sigma'^2_{i+2} < \ldots$ sein müssen. Reihen, für welche solche Beziehungen bestehen, gehören zur "Z-Gruppe."

Und umgekehrt, wenn diese linken Seiten in (33) und (33a), von i angefangen, alle < 0 sind, so haben wir, im Gegenteil, $E\sigma_i'^2 > E\sigma'^2_{i+1} > E\sigma'^2_{i+2} > \ldots$ Reihen mit solchen Beziehungen zwischen ihren Streuungen gehören zur "G-Gruppe."

Betrachten wir anfangs den zweiten Fall und begnügen wir uns mit der Untersuchung nur derjenigen Reihen, für welche *von Anfang an*

$$E\sigma_0'^2 \overset{\ddagger}{+} > E\sigma_1'^2 > E\sigma_2'^2 > E\sigma_3'^2 > \ldots.$$

(Im Verhältnis zu den Streuungen der "R-Gruppe" werden also die Streuungen dieser Gruppe mit jeder Differenz immer geringer.) Dann können wir zum System (33a) noch die fehlende Ungleichung $-r_1 < 0$ hinzufügen und erhalten dann: $r_1 > 0$; $r_1 - r_2 > 0$; $r_1 - r_2 > \frac{3}{2}(r_1 - 2r_2 + r_3)$, u.s.w. Die Ungleichungen werden mit wachsenden i immer komplizierter und weniger durchsichtig. Um zu einfacheren Formeln zu gelangen, wollen wir jetzt noch eine weitere begrenzende Bedingung einführen: die linken Seiten des Systems (33a) werden nämlich *a fortiori* < 0 sein, wenn alle $\nabla^{(i)} r_1 > 0$ (bei $i = 1, 2, 3, \ldots k$). Und dann kommen wir zum System:

$$r_1 > 0;\ r_1 > r_2;\ \nabla' r_1 > \nabla' r_2;\ \nabla'' r_1 > \nabla'' r_2, \ldots \nabla^{k-1} r_1 > \nabla^{k-1} r_2.$$

Dieses System ist viel handlicher und kann durch verschiedene "Modelle" dargestellt werden. Am einfachsten würde wohl die Annahme sein: $r_1 = p$, $r_2 = p^2$, $r_3 = p^3, \ldots r_i = p^i$. In diesem Falle ergiebt es sich nämlich, daß $\nabla^i r_i = p(1-p)^i$, und dieser Ausdruck bleibt immer positiv, solange $1 > p > 0$.

* Besäßen wir Kenntnis über den genauen Wert von Es, so könnten wir leicht, falls schon $\frac{E\sum_{i=1}^{N}(s_i - Es)^2}{N} = \frac{E\sum_{i=1}^{N-1}(\Delta' s)^2}{2(N-1)}$, die fehlende Gleichung $r_1 = 0$ erhalten und das System (33) eindeutig auflösen. Über eine solche Kenntnis verfügen wir aber nur in den seltensten Fällen. Aus der Gleichung $\frac{E\sum_{i=1}^{N}(s_i - s_{(N)})^2}{N-1} = \frac{E\sum_{i=1}^{N-1}(\Delta' x_i)^2}{2(N-1)}$, wo $s_{(N)} = \frac{1}{N}\sum_{i=1}^{N} s_i$, erhalten wir die fehlende Gleichung *nicht*, da in ihr, dank der Einführung von $s_{(N)}$, alle r_i von r_1 bis r_{N-1} auftreten werden. Und wenn man sogar bis zur $(N-1)$-ten Differenz vorgeht, so erweist es sich, daß diese Gleichung schon im vollen System (33) enthalten ist, also nichts neues bietet.

Direkt ist kein einziges r_i zu bestimmen, da es keine einzige empirische Formel gibt und *geben kann*, deren mathematische Erwartung genau r_i ausmachen würde. Vergl. hierüber noch unten, eine Anmerkung in § (12).

† Ebenda, S. 142—143.

‡ Siehe Form. (9).

Andererseits, würden wir auch in dem Falle zu einfachen Formeln gelangen, wenn wir wieder annehmen, daß die Reihe r durch eine gewisse Parabel i-ter Ordnung dargestellt werden könne. Dann würden alle endlichen Differenzen von der Ordnung $(i+1)$ aufwärts gleich 0 sein. Verschwinden, zum Beispiel, schon die zweiten Differenzen, so erhalten wir folgendes leicht faßliches System:

$$r_1 - r_2 = r_2 - r_3 = r_3 - r_4 = \ldots = r_{k-1} - r_k > 0.$$

Verschwinden erst die 3-tten Differenzen, so kommen wir zum System:

$$\frac{r_1 + r_3}{2} - r_2 = \frac{r_2 + r_4}{2} - r_3 = \ldots = \frac{r_{k-2} + r_k}{2} - r_{k-1} > 0, \text{ u.s.w.}$$

Solch ein Verhalten würde für eine "zufällige" Variable, als welche wir die "restliche Komponente" ansehen, wenig wahrscheinlich sein, *könnte aber sehr wohl ein Gegenstück für die Hypothese einer parabolischen Beschaffenheit der "glatten Komponente" abgeben.*

Schließlich sei noch bemerkt, daß auch Yule's Sinus-Reihen mit der Formel $u_i = A \sin\left(2\pi \frac{t+\tau+ih}{T}\right)$ im Zwischenraum $0 \leqq h \leqq \frac{1}{4}T$ und $T \geqq h \geqq \frac{3}{4}T$ ebenfalls zur "G-Gruppe" gehören.

Was nun den 3-tten Fall anbetrifft, das heißt, die Bedingungen für eine solche "restliche Komponente," welche beim endlichen Differenzieren rascher wächst, als es eine Reihe mit gegenseitig unabhängigen oder uniform verbundenen Versuchen zuläßt ("Z-Gruppe"), so können, wie gesagt, zu ihrer Untersuchung ebenfalls die Formeln des Systems (33a) zugezogen werden, indem hier vor 0 überall das Zeichen $>$ gesetzt wird. Und ebenfalls *a fortiori* kann man auch folgende Bedingungen aufstellen:

$$r_1 < 0,\ r_1 < r_2,\ \nabla' r_1 < \nabla' r_2,\ \nabla'' r_1 < \nabla'' r_2, \ldots \nabla^{k-1} r_1 < \nabla^{k-1} r_2.$$

Diesem System würde, zum Beispiel, eine jede Reihe s genügen, bei welcher alle r_i bei ungeradem i negativ und bei geradem i positiv sein würden. Zum Beispiel:

(1) $r_1 = p,\ r_2 = p^2,\ r_3 = p^3, \ldots r_i = p^i$, wenn $0 > p \geqq -1$; oder:

(2) $r_1 = A - a,\ r_2 = A + a, \ldots r_i = A + (-1)^i a$, wenn $a > |A|$ und $|A| + a < 1$; oder:

(3) $r_i = A + (-1)^i (a + ih)$, wobei jedoch $|kh| < a > 0$ sein muß, wenn h negativ ist, und außerdem $|A| + a + |kh| < 1$, u.s.w.

In diese Klasse gehören auch Yule's kurzperiodische Sinus-Reihen.

(11) Wenn wir nun von diesen wahrscheinlichkeitstheoretischen Schemata zur statistischen Wirklichkeit zurückkehren, wo, in der Regel, weder r_i noch $E\sigma_i'^2$ gegeben sind, so ist es am vorteilhaftesten, von der Betrachtung der Differenzen $\sigma'^2_{i+1} - \sigma_i'^2$, $\sigma'^2_{i+2} - \sigma'^2_{i+1}$, $\ldots\ \sigma_k'^2 - \sigma'^2_{k-1}$, auszugehen. Können wir an Hand der Formeln der Tabelle II feststellen, daß diese Differenzen relativ noch so klein sind, daß man *modo Bernoulliano* das Bestehen der Gleichungen $E\sigma_i'^2 = E\sigma'^2_{i+1} = \ldots$ annehmen darf, so kann man, auf Grund des im vorhergehenden Paragraphen gesagten,—mit einer Wahrscheinlichkeit, die desto größer

wird, je weiter der Zwischenraum zwischen i und k,—annehmen, daß die i-ten endl. Differenzen der Reihe $u_1, u_2, \ldots u_N$ nur lineare Funktionen einer zur "R-Gruppe" gehörigen Komponente s enthalten. Man kann dabei mit einer ziemlichen Sicherheit, wenigstens für größere N, erwarten, daß die Glieder derselben entweder unabhängig von einander oder, im schlimmsten Fall, miteinander nur uniform verbunden sind. Bleiben die Differenzen der Streuungen bis $\sigma_k'^2 - \sigma'^2_{k-1}$ im Verhältniss zu ihren mittleren Fehlern groß und dabei negativ, so kann man noch mit höheren Differenzen sein Glück versuchen. Sind sie, im Gegenteil, groß und positiv, so bedeutet das die wahrscheinliche Anwesenheit einer Komponente von der "Z-Gruppe."

Es muß hierbei noch folgendes bemerkt werden. Gehört die Reihe s zur "R-Gruppe," so besitzt die Reihe $\Delta' s_1, \Delta' s_2, \ldots$ schon die Eigenschaften einer "Z-Gruppe," und je höher k, die Ordnung der Differenz, desto mehr werden diese "Z"-Eigenschaften hervortreten. Besteht nun jedes Glied der Reihe: $s_1, s_2, s_3, \ldots$ selbst aus einer Summe von 2 Komponenten: einer "R"-artigen und einer "Z"-artigen, so kann die "schädliche" Wirkung der letzteren in den meisten Fällen sich nur in den wenigen ersten Differenzen bemerkbar machen. Das Maximum der "Schädlichkeit" erreicht eine Komponente der "Z-Gruppe" bei

$$r_1 = -1,\ r_2 = +1, \ldots r_i = (-1)^i.$$

Nun ist für eine solche Reihe

$$\frac{E\sum_{i=1}^{N-k}(\Delta_i^k)^2}{N-k} = 4^k \mu_2,$$

während eine von der "R-Gruppe" bloß $\frac{2k!}{k!\,k!}\,\mu_2$ ergeben würde, oder, angenähert, $\frac{4^k}{\sqrt{\pi k}}\,\mu_2$ (vergl. Form. 11 a). Für $k = 6$ würde das eine $4\frac{100}{231}$ × fache Vergrößerung des "relativen Einflusses" der *maximalen* "Z"-Komponente bedeuten. Eine weitere Verdoppelung desselben tritt aber nur bei $k = 24$ ein. Ist daher, zum Beispiel, an der Differenz $\sigma_6'^2 - \sigma_5'^2$ die Wahrscheinlichkeit des Vorhandenseins einer "Z-Gruppen"-Komponente nicht festzustellen, so ist es auch verhältnismäßig wenig wahrscheinlich, daß die weiteren Differenzen eine solche Komponente zum Vorschein treten lassen könnten (wenn, natürlich, in der 6-ten Differenz die Komponente "G" schon eliminiert ist). Diese "Z"-Komponente existiert, offenbar, überhaupt nicht oder ist im Verhältnis zur "R"-Komponente nicht bedeutend.

Es entsteht nun die Frage: sind die möglichen Ergebnisse einer Untersuchung der Streuungen der Differenzen einer gegebenen Reihe auch genügend wichtig, um die hierbei erforderliche Rechenarbeit zu rechtfertigen? hat die Unterscheidung der Gruppen "Z," "R" und "G" irgendeinen Erkenntniswert auch außerhalb der Anwendung der Differenzenmethode?

(*Ein Schlußartikel folgt.*)

ON THE MEANS AND SQUARED STANDARD-DEVIATIONS OF SMALL SAMPLES FROM ANY POPULATION.

By A. E. R. CHURCH, M.A., B.Sc.

Part I. Introductory.

The theory of sampling, both as to mean and standard-deviation, has been known with completeness for some considerable time, provided the sampled population follows the *normal* law. That the distribution of means follows the normal law whatever the size of the sample, provided the sampling is from normal material, has been known from the early days of probability theory and the moment relations of the distribution of means of such samples were determined by Tchebycheff many years ago. The distribution of standard-deviations of samples from normal material has also been fully worked out both for small and large samples, and actual experimental verification of the theoretical formulae obtained was provided by "Student" in *Biometrika*, Vol. VI. pp. 1—25.

While the work indicated above has rounded off the case of large or small samples from a normal population, both from the point of view of their means and their standard-deviations, it gives us no ability to deal with the case of samples from material which is definitely skew in character.

However, relations of the type:

$$ {}_1M_1' = \bar{\mu}_1' \quad \ldots\ldots(1), \qquad {}_1B_1 = \frac{\bar{\beta}_1}{N} \quad \ldots\ldots(5), $$

$$ {}_1M_2 = \frac{\bar{\mu}_2}{N} \quad \ldots\ldots(2), \qquad {}_1B_2 - 3 = \frac{\bar{\beta}_2 - 3}{N} \quad \ldots\ldots(6), $$

$$ {}_1M_3 = \frac{\bar{\mu}_3}{N^2} \quad \ldots\ldots(3), \qquad \Sigma_1 = \frac{\bar{\sigma}}{N^{\frac{1}{2}}} \quad \ldots\ldots(7), $$

$$ {}_1M_4 = \frac{\bar{\mu}_4}{N^3} + \frac{3(N-1)}{N^3} \cdot \bar{\mu}_2^2 \ldots(4), $$

where ${}_1M_1'$, ${}_1M_2$, ${}_1M_3$, ${}_1M_4$, ${}_1B_1$, ${}_1B_2$ and Σ_1 are the mean, second, third and fourth moment-coefficients, first two betas and standard-deviation respectively of the distribution of means of samples of N, whilst $\bar{\mu}_1'$, $\bar{\mu}_2$, $\bar{\mu}_3$, $\bar{\mu}_4$, $\bar{\beta}_1$, $\bar{\beta}_2$ and $\bar{\sigma}$ are the corresponding constants of the sampled population, follow from Tchebycheff's work and enable us to obtain some appreciation of the distribution of means of samples of any size from any population. It is clear from them that, if the deviation of the sampled population from normality be not very great, the distribution of means of samples of N itself will not diverge widely from normality. In fact for $\beta_1 \leqslant 0{\cdot}2$ and $\bar{\beta}_2 - 3 \leqslant 0{\cdot}4$ it is to be expected that, even for $N = 10$, a normal curve will give a really excellent fit to an experimental distribution of means.

As regards theoretical distributions of means of samples of N obtained directly without the use of moments from a given sampled population, when this population is a Type III Pearson Curve the corresponding distribution of means has been deduced in Part II by direct integration, the sampled population being considered "infinite" in all its categories *.

The deduction of β-relations for the distribution of σ^2 of samples drawn from any sampled population infinite in all its categories is due to Tchouproff †. At first sight these relations place the distributions of means and σ^2 of samples from non-normal material on the same footing; that is the constants of these distributions can be computed and frequency curves to represent them can be determined. But as far as calculation is concerned, the two cases are on somewhat different footings. Tchouproff's formulae are very lengthy, one of them having 37 terms in it; but, what is much more troublesome in practice, the evaluation of the formula for the 4th moment-coefficient of the distribution of σ^2 of samples involves a computation of the moments of the sampled population up to and including the 8th. Again Tchouproff's formulae are absolute, not approximate, but it would be wrong to suppose that, for small and moderately small samples, the terms in the higher inverse powers of N can be neglected summarily in comparison with the terms in the lower inverse powers of N. For such values of the moment-coefficients as obtain for moderate skewness of the sampled population and moderately small samples, I find there is not such a rapid convergence of the Tchouproff terms as to warrant the general neglect of the terms in the higher inverse powers of N. This is a practical illustration of the theorem developed by Isserlis, namely that the moment-coefficients of the distribution of σ^2 of samples from an infinite population do obey, when the sample is large enough, the normal law, but the sample must be very large before we can look upon the distribution of even the 4th moment-coefficient as normal.

Now the computation of the 8th moment-coefficient of the sampled population, as required by Tchouproff's formulae, can be carried out in a special experimental investigation like this, but can hardly be demanded in everyday statistical practice. Accordingly I looked round for a method of determining the higher betas without direct computation and it seemed desirable to ascertain whether, assuming the applicability of Pearson's general frequency curves which provide finite difference relations ‡ between successive betas, it is possible to determine the higher betas of the sampled population from the lower ones by these formulae with sufficient accuracy to obtain good fits to the experimental distributions of σ^2 of samples obtained from the skew populations employed.

* The term "infinite sampled population" has a quite simple and definite meaning for the statistician, although it seems to have troubled some mathematicians recently (see Burnside: *Camb. Phil. Soc. Proc.* Vol. XXII, pp. 726—7). I indicate its significance below.

† *Biometrika*, Vol. XII. pp. 193—194. For the correction of Tchouproff's formulae in the case of the 4th moment-coefficient by the present writer, see *Biometrika*, Vol. XVII. pp. 79—83.

‡ Tables of β_3, β_4, β_5 and β_6 in terms of β_1 and β_2 are given in *Tables for Statisticians*, pp. 78—79, but as some errors occur they have been recalculated by Professor K. Yasukawa: see pp. 268—275 of the present issue of this Journal.

It is well at this point to draw attention to an important matter. The population from which the samples are taken will be in general itself a sample for a much larger population, and hence its higher moment-coefficients may differ from their "true" values by very considerable variations, by even 40 or 50 per cent. Accordingly on this ground alone we should not be justified—except for the special purpose of testing our formulae experimentally—in laying any marked stress on the exact values of these higher betas of the sampled population.

This method of obtaining rapidly the higher betas of the sampled population will be shown to be successful and thus, expressing Tchouproff's formulae in terms of the betas of the sampled population, we can use for the higher betas their values in terms of the lower ones as obtained by the difference relations.

While formulae for the moment-coefficients of the distributions of means and σ^2 of samples of N from any sampled population may now be said to be established and confirmed experimentally, it must yet be remarked that they are dependent on the sampled population being indefinitely large in all its categories compared with the size of the sample. In practice this means that when any individual is drawn in actual sampling, the chance that the next one drawn will belong to a definite category of the sampled population must be quite unaffected by this previous "draw," or otherwise individuals must be drawn singly and replaced before the next is drawn. This method is generally referred to as sampling from an "infinite" population. If the whole sample be drawn before replacement, the frequency of draws from a single category of the sampled population will no longer follow a binomial but a hypergeometrical distribution. This method is generally referred to as sampling from a "finite" population.

Turning now to the consideration of sampling from a finite population, the moment-coefficients of the distribution of means of samples of N have been known for some time, although it is not clear from whom they originated, whilst Dr J. Splawa-Neyman* has given the mean and 2nd moment-coefficient of the distribution of σ^2 of such samples. However, he did not give formulae for the 3rd and 4th moment-coefficients of the distribution of σ^2 of samples, which moment-coefficients are required if the distribution is to be represented by a Pearson or other frequency curve. These formulae proved very laborious and difficult to obtain, and when obtained were found to be extremely unwieldy and complicated, but they are given together with an outline of the method of obtaining them in the last part of this paper.

This very considerable piece of work has apparently placed the discussion of the distributions of means and σ^2 of samples of N from a finite population on the same basis as that given for the corresponding distributions obtained from an infinite population. This is more or less true for the distribution of means of samples, for like the corresponding distribution from an infinite population, it can be represented by a Pearson curve obtained from the values of the moment-coefficients

* *Biometrika*, Vol. XVII. p. 472. In this paper Dr Neyman also gives the moment-coefficients of the distribution of means of samples.

given by their formulae, and moreover, even when the sampled population is very skew and as small as 36, the distribution shows a remarkable approach to normality.

In considering the distribution of σ^2 of samples of N from a finite population, the formulae for the moment-coefficients being so unwieldy, it is found that even moderate-sized sampled populations must be treated as infinite; that is, Tchouproff's formulae must be used in lieu of the proper formulae for these moment-coefficients, if the labour involved in the representation of the distribution is to be kept within limits that are reasonable even for an investigation like this. In practical statistics the labour involved would undoubtedly prohibit in all cases the use of these proper formulae for the moment-coefficients.

These formulae could be used for very small sampled populations like the binomial considered below, but judging from this case, it is likely that the resulting frequency curve would in no way represent even the general trend of the frequency groups of an actual distribution of σ^2 of samples obtained from the population. This distribution of σ^2 of samples in the case considered proved to be a discrete frequency with widely varying intervals between the separate frequency groups, and was moreover tri-modal. This rather unprecedented and startling result, coupled with the unwieldiness that is certain to appear in any attempt to find formulae for the higher moment-coefficients of the distribution of σ^2 of samples, means that, at present, the representation of such a distribution by a smooth frequency curve is practically impossible, if the sampled population be very small.

For the true application of all this theory, too much emphasis cannot be laid on the fact that the samples in any actual case must be truly "random" in character. The recording of the individual, its return to the "bag" and the thorough shuffling between each draw involve, in the case of 10,000 or 20,000 draws, a very great expenditure of time and labour. At the same time, in my experience and that of other workers in the Biometric Laboratory, it is impossible with many hundreds of marked tickets to ensure the returned ticket being equally likely to be anywhere in the bag or bowl, and so equally likely to be drawn next time as any other ticket. In fact, working with many tickets, perfect shuffling seems a practical impossibility.

However, when the sampling is conducted on the basis of an infinite sampled population, the most satisfactory method seems to be to procure one good random sample, such as that obtained by Mr L. H. C. Tippett, and to apply this to all samplings of this character. This method, which is described later, has also the great advantage of comparative speed and ease of operation, and I wish at this stage to thank Mr Tippett for the loan of his set of random numbers, for it is undoubtedly a boon when sampling from an infinite population.

When the sampling is made on the basis of a finite sampled population, however, such a set of random numbers is at present of no avail. Consequently, whilst discussing the distributions of means and σ^2 of samples of N from such populations, material other than tickets was used in order to find a way of minimising the difficulties of shuffling.

When the sampled population, though finite, is large, quite approximately symmetrical and homogeneous material like small commercial coloured glass beads has been found to overcome the difficulties of shuffling to the extent necessary to produce good agreement between theory and practice. When however the sampled population is quite small, satisfactory results can only be obtained by using material like coloured marbles, which approximate very closely indeed to the obvious ideal of complete symmetry of shape and homogeneity of size and weight.

Turning now to the consideration of the experimental material used, it should be remarked that the number of significant figures given for the moment-coefficients is only adopted in order to obtain the higher betas with the accuracy needful for their use in Tchouproff's formulae*. They can, of course, be dropped as soon as the two betas of the distribution of means or of second moments of samples, as the case may be, have been reached. The two sampled populations used in the discussion of samples from an infinite population are the following:

(A) *Barometric Readings taken over a period of eleven Years at Laudale in Scotland.*

Total Frequency 4011.

Central Bar. Height in ins.	*27·9*	*28·0*	*28·1*	*28·2*	*28·3*	*28·4*	*28·5*	*28·6*	*28·7*	*28·8*	*28·9*	*29·0*	*29·1*	*29·2*	*29·3*
Frequency	1	1	0	0	0	2	2	4	8	12	32	47	72	87	125
Central Bar. Height in ins.	*29·4*	*29·5*	*29·6*	*29·7*	*29·8*	*29·9*	*30·0*	*30·1*	*30·2*	*30·3*	*30·4*	*30·5*	*30·6*	*30·7*	*30·8*
Frequency	187	255	304	346	394	380	400	361	376	266	182	94	50	21	2

The Constants for this distribution were found to be as follows:

[Working Origin 29·7 ins. of Bar. Height. Working Unit 0·1 in. of Bar. Height.]

$\nu_1' =$ 1·580,155, $\quad \nu_5' =$ 1,416·561,456,

$\nu_2' =$ 17·255,547, $\quad \nu_6' =$ 57,788·534,530,

$\nu_3' =$ 49·231,613, $\quad \nu_7' = -$ 155,697·948,891,

$\nu_4' =$ 759·882,324, $\quad \nu_8' =$ 7,868,534·826,228.

Mean = 29·858,0155 ins. of Bar. Height.

$\nu_2 =$ 14·758,657, $\quad \nu_6 =$ 70,469·386,836,

$\nu_3 = -$ 24·676,758, $\quad \nu_7 = -$ 818,237·881,078,

$\nu_4 =$ 688·517,798, $\quad \nu_8 =$ 13,875,641·654,318,

$\nu_5 = -$ 3,999·253,754, $\quad \sigma =$ 0·384,1700 in. of Bar. Height.

$\beta_1 =$ 0·189,424, $\quad \beta_2 =$ 3·160,978, $\quad \beta_3 =$ 2·080,078,

$\beta_4 =$ 21·920,980, $\quad \beta_6 =$ 292·459,489.

The Skewness is ·237.

* Some of the powers and products of moment-coefficients have to be multiplied by numerical coefficients of the order of 1300.

(B) *A Graduated Frequency Distribution obtained from a smooth Frequency Curve given in* Biometrika, Vol. XI. p. 400.

Total Frequency 10,000.

Central Abscissa of Group	−14	−13	−12	−11	−10	−9	−8	−7	−6	−5	−4	−3	−2
Frequency	1	2	4	8	15	27	47	78	125	191	282	399	541
Central Abscissa of Group	−1	0	1	2	3	4	5	6	7	8	9	10	11
Frequency	700	861	1005	1101	1127	1064	910	691	451	241	98	27	4

The Constants for this distribution were found to be:

$\nu_1' =$ 1·8518, $\nu_5' =$ 2,274·2528,
$\nu_2' =$ 16·1684, $\nu_6' =$ 35,855·6624,
$\nu_3' =$ 55·8272, $\nu_7' =$ 95,052·5432,
$\nu_4' =$ 628·5896, $\nu_8' =$ 2,676,103·2296.

Mean = 1·8518.

$\nu_2 =$ 12·739,237, $\nu_6 =$ 38,480·027,968,
$\nu_3 = -$ 21·294,481, $\nu_7 = -$ 329,631·122,867,
$\nu_4 =$ 512·453,377, $\nu_8 =$ 4,368,582·223,120,
$\nu_5 = -$ 2,571·063,549, $\sigma =$ 3·569,207.

$\beta_1 =$ 0·219,333, $\beta_2 =$ 3·157,676, $\beta_3 =$ 2·078,769,
$\beta_4 =$ 18·612,518, $\beta_6 =$ 165·869,619.

The Skewness is ·263.

Since for the purpose of sampling the material must be arranged in groups and the individuals of one group all given the *same* class-index, it is not only unnecessary, but even erroneous, to apply Sheppard's grouping corrections to the moment-coefficients, before using them to determine the distributions of means and squared standard-deviations of samples. Consequently in the preceding values, and throughout this paper, the corrections for grouping have not been used, the betas being calculated directly from the raw moments (ν's) transferred to the mean.

The sets of samples taken from these two populations (A) and (B) are indicated in the following, the constants of the corresponding distributions of means and squared standard-deviations (second moments) of samples being given at the same time.

I. 1000 Samples of 10 from Population (A), the sampling being on the basis of an "infinite" population.

Method of Sampling, that of drawing single tickets with return from a bag.

(*a*) *Frequency Distribution of Means of Samples.*

Total Frequency 1000.

Bar. Height in ins.	29·275–	29·325–	29·375–	29·425–	29·475–	29·525–	29·575–	29·625–	29·675–	29·725–	29·775–
Frequency	1	1	0	7	9	18	22	52	83	125	132
Bar. Height in ins.	29·825–	29·875–	29·925–	29·975–	30·025–	30·075–	30·125–	30·175–	30·225–	30·275–	30·325–
Frequency	158	145	101	63	37	29	13	3	0	0	1

The Constants were found to be:

[Working Origin = 29·8 ins. of Bar. Height. Working Unit = 0·05 in. of Bar. Height.]

$\nu_1' =$ 0·741, Mean = 29·837,050 ins. of Bar. Height.
$\nu_2' =$ 8·149, $\nu_2 =$ 7·599,919, $\sigma =$ 0·137,840 in. of Bar. Height,
$\nu_3' =$ 14·367, $\nu_3 = -$ 2·934,489, $\beta_1 =$ 0·019,617,
$\nu_4' =$ 213·217, $\nu_4 =$ 196·575,508, $\beta_2 =$ 3·403,388.

The Skewness is ·057.

(*b*) *Frequency Distribution of σ^2's of Samples.*

Total Frequency 1000.

Value of σ^2	1·5–	2·5–	3·5–	4·5–	5·5–	6·5–	7·5–	8·5–	9·5–	10·5–	11·5–	12·5–	13·5–
Frequency	1	3	9	18	25	39	48	60	56	64	51	71	52
Value of σ^2	14·5–	15·5–	16·5–	17·5–	18·5–	19·5–	20·5–	21·5–	22·5–	23·5–	24·5–	25·5–	26·5–
Frequency	65	47	53	46	32	40	43	19	25	16	16	14	16
Value of σ^2	27·5–	28·5–	29·5–	30·5–	31·5–	32·5–	33·5–	34·5–	35·5–	36·5–	37·5–	38·5–	39·5–
Frequency	13	5	9	7	3	2	5	3	0	4	2	2	1
Value of σ^2	40·5–	41·5–	42·5–	43·5–	44·5–	45·5–	46·5–	47·5–	48·5–	49·5–	50·5–	51·5–	52·5–
Frequency	1	1	1	1	0	0	2	1	1	2	2	0	3

The Constants were found to be:

[Working Origin = 14.]

$\nu_1' =$ 1·848, Mean = 15·848, $\beta_1 =$ 2·149,615,
$\nu_2' =$ 64·578, $\nu_2 =$ 61·162,896, $\beta_2 =$ 6·560,572,
$\nu_3' =$ 1,046·712, $\nu_3 =$ 701·313,792, $\sigma =$ 7·820,671.
$\nu_4' =$ 30,991·482, $\nu_4 =$ 24,542·441,607,

The Skewness is ·643.

II. 1000 Samples of 10 from Population (A), the sampling being on the basis of a "finite" population.

Method of Sampling, that of drawing tickets from a bag, complete samples being drawn before return to bag.

(*a*) *Frequency Distribution of Means of Samples.*

Total Frequency 1000.

Bar. Height in ins.	*29·445–*	*29·485–*	*29·525–*	*29·565–*	*29·605–*	*29·645–*	*29·685–*	*29·725–*	*29·765–*	*29·805–*
Frequency	1	4	11	16	21	37	69	94	128	120
Bar. Height in ins.	*29·845–*	*29·885–*	*29·925–*	*29·965–*	*30·005–*	*30·045–*	*30·085–*	*30·125–*	*30·165–*	*30·205–*
Frequency	115	129	89	77	45	26	11	5	1	1

The Constants were found to be:

[Working Origin = 29·825 ins. of Bar. Height. Working Unit = 0·04 in. of Bar. Height.]

$\nu_1' = 0{\cdot}475$, Mean = 29·844,000 ins. of Bar. Height, $\beta_1 = 0{\cdot}013{,}218$,
$\nu_2' = 9{\cdot}459$, $\nu_2 = 9{\cdot}233{,}375$, $\beta_2 = 2{\cdot}907{,}286$,
$\nu_3' = 10{\cdot}039$, $\nu_3 = -3{\cdot}225{,}731$, $\sigma = 0{\cdot}121{,}546$ in.
$\nu_4' = 254{\cdot}283$, $\nu_4 = 247{\cdot}861{,}301$, of Bar. Height.

The Skewness is ·063.

(*b*) *Frequency Distribution of σ^2's of Samples.*

Total Frequency 1000.

Value of σ^2	*0·5–*	*1·5–*	*2·5–*	*3·5–*	*4·5–*	*5·5–*	*6·5–*	*7·5–*	*8·5–*	*9·5–*	*10·5–*	*11·5–*
Frequency	2	4	5	19	24	35	47	56	83	58	82	65
Value of σ^2	*12·5–*	*13·5–*	*14·5–*	*15·5–*	*16·5–*	*17·5–*	*18·5–*	*19·5–*	*20·5–*	*21·5–*	*22·5–*	*23·5–*
Frequency	54	67	57	63	44	34	33	19	26	17	29	11
Value of σ^2	*24·5–*	*25·5–*	*26·5–*	*27·5–*	*28·5–*	*29·5–*	*30·5–*	*31·5–*	*32·5–*	*33·5–*	*34·5–*	*35·5–*
Frequency	10	7	9	7	6	2	4	6	3	1	1	2
Value of σ^2	*36·5–*	*37·5–*	*38·5–*	*39·5–*	*40·5–*	*41·5–*	*42·5–*	*43·5–*	*44·5–*			
Frequency	1	3	1	1	1	0	0	0	1			

The Constants were found to be:

[Working Origin = 12.]

$\nu_1' =$ 1·915, Mean = 13·915, $\beta_1 =$ 1·113,857,

$\nu_2' =$ 46·419, $\nu_2 =$ 42·751,775, $\beta_2 =$ 4·684,768,

$\nu_3' =$ 550·903, $\nu_3 =$ 298·271,317, $\sigma =$ 6·538,484.

$\nu_4' =$ 11,801·307, $\nu_4 =$ 8,562·417,905,

The Skewness is ·524.

III. 1000 Samples of 10 from Population (A), the sampling being on the basis of an "infinite" population.

Method of Sampling, that by aid of Tippett's Table of Random Numbers*.

(*a*) *Frequency Distribution of Means of Samples.*

Total Frequency 1000.

Bar. Height in ins.	*29·425–*	*29·475–*	*29·525–*	*29·575–*	*29·625–*	*29·675–*	*29·725–*	*29·775–*
Frequency	2	3	7	20	44	67	109	158
Bar. Height in ins.	*29·825–*	*29·875–*	*29·925–*	*29·975–*	*30·025–*	*30·075–*	*30·125–*	*30·175–*
Frequency	146	163	123	85	44	20	8	1

The Constants were found to be:

[Working Origin = 29·85 ins. of Bar. Height. Working Unit = 0·05 in. of Bar. Height.]

$\nu_1' =$ 0·063, Mean = 29·853,150 ins. of Bar. Height, $\beta_1 =$ 0·019,694,

$\nu_2' =$ 5·889, $\nu_2 =$ 5·885,031, $\beta_2 =$ 2·929,922,

$\nu_3' = -$ 0·891, $\nu_3 = -$ 2·003,521, $\sigma =$ 0·121,295 in.

$\nu_4' =$ 101·109, $\nu_4 =$ 101·473,726, of Bar. Height.

The Skewness is ·075.

* A description of this method is given at the end of this part.

(*b*) *Frequency Distribution of σ^2's of Samples.*

Total Frequency 1000.

Value of σ^2	1·5–	2·5–	3·5–	4·5–	5·5–	6·5–	7·5–	8·5–	9·5–	10·5–	11·5–	12·5–	13·5–	14·5–
Frequency	2	10	17	27	45	57	55	58	75	73	79	63	48	62
Value of σ^2	15·5–	16·5–	17·5–	18·5–	19·5–	20·5–	21·5–	22·5–	23·5–	24·5–	25·5–	26·5–	27·5–	28·5–
Frequency	59	42	43	29	24	27	15	13	15	13	11	6	4	4
Value of σ^2	29·5–	30·5–	31·5–	32·5–	33·5–	34·5–	35·5–	36·5–	37·5–	38·5–	39·5–	40·5–	41·5–	42·5–
Frequency	5	1	3	0	0	2	2	1	1	2	1	0	1	0
Value of σ^2	43·5–	44·5–	45·5–	46·5–	47·5–	48·5–	49·5–	50·5–	51·5–	52·5–	53·5–	54·5–	55·5–	
Frequency	0	0	0	0	1	0	0	1	0	2	0	0	1	

The Constants were found to be:

[Working Origin = 12.]

$\nu_1' =$ 1·725, Mean = 13·725, $\beta_1 = 2{\cdot}659{,}598$,

$\nu_2' =$ 48·805, $\nu_2 =$ 45·829,375, $\beta_2 = 8{\cdot}520{,}217$,

$\nu_3' =$ 748·269, $\nu_3 =$ 505·969,031, $\sigma = 6{\cdot}769{,}732$.

$\nu_4' = 22{,}213{\cdot}549$, $\nu_4 = 17{,}895{\cdot}282{,}136$,

The Skewness is ·532.

IV. 1000 Samples of 10 from Population (B), the sampling being on the basis of an "infinite" population.

Method of Sampling, that by aid of Tippett's Table of Random Numbers.

(*a*) *Frequency Distribution of Means of Samples.*

Total Frequency 1000.

Value of the Mean	−2·49–	−1·99–	−1·49–	−0·99–	−0·49–	0·01–	0·51–	1·01–	1·51–
Frequency	1	2	6	16	44	53	129	152	188
Value of the Mean	2·01–	2·51–	3·01–	3·51–	4·01–	4·51–	5·01–	5·51–	
Frequency	157	121	78	33	15	4	0	1	

The Constants were found to be:

[Working Origin = 1·76. Working Unit = 0·5.]

$\nu_1' = -$ 0·011,	Mean = 1·7545,	$\beta_1 = 0·010,318$,
$\nu_2' =$ 5·081,	$\nu_2 =$ 5·080,879,	$\beta_2 = 3·095,297$,
$\nu_3' = -$ 1·331,	$\nu_3 = -$ 1·163,329,	$\sigma =$ 1·127,041.
$\nu_4' =$ 79·961,	$\nu_4 =$ 79·906,126,	

The Skewness is ·048.

(*b*) *Frequency Distribution of σ^2's of Samples.*

Total Frequency 1000.

Value of σ^2	*0·5–*	*1·5–*	*2·5–*	*3·5–*	*4·5–*	*5·5–*	*6·5–*	*7·5–*	*8·5–*	*9·5–*	*10·5–*	*11·5–*	*12·5–*	*13·5–*
Frequency	1	1	18	34	66	67	74	80	62	85	71	61	67	51
Value of σ^2	*14·5–*	*15·5–*	*16·5–*	*17·5–*	*18·5–*	*19·5–*	*20·5–*	*21·5–*	*22·5–*	*23·5–*	*24·5–*	*25·5–*	*26·5–*	*27·5–*
Frequency	48	34	33	34	26	22	8	14	7	7	5	4	2	5
Value of σ^2	*28·5–*	*29·5–*	*30·5–*	*31·5–*	*32·5–*	*33·5–*	*34·5–*	*35·5–*	*36·5–*	*37·5–*	*38·5–*	*39·5–*	*40·5–*	
Frequency	0	5	1	2	2	1	1	0	0	0	0	0	1	

The Constants were found to be:

[Working Origin = 10.]

$\nu_1' =$ 1·609,	Mean = 11·609,	$\beta_1 = 1·136,733$,
$\nu_2' =$ 34·793,	$\nu_2 =$ 32·204,119,	$\beta_2 = 4·683,644$,
$\nu_3' =$ 354·463,	$\nu_3 =$ 194·848,209,	$\sigma =$ 5·674,867.
$\nu_4' =$ 6,618·413,	$\nu_4 =$ 4,857·431,837,	

The Skewness is ·539.

V. 1000 Samples of 10 from Population (A), the sampling being on the basis of a "finite" population.

Method of Sampling, that of drawing coloured beads from a bowl, complete samples being drawn before return to bowl.

(*a*) *Frequency Distribution of Means of Samples.*

Total Frequency 1000.

Bar. Height in ins.	*29·425–*	*29·475–*	*29·525–*	*29·575–*	*29·625–*	*29·675–*	*29·725–*	*29·775–*
Frequency	2	2	7	25	34	67	92	136
Bar. Height in ins.	*29·825–*	*29·875–*	*29·925–*	*29·975–*	*30·025–*	*30·075–*	*30·125–*	over *30·175*
Frequency	170	154	127	84	54	29	13	4

The Constants were found to be:

[Working Origin = 29·85 ins. of Bar. Height. Working Unit = 0·05 in. of Bar. Height.]

$\nu_1' =$ 0·274, Mean = 29·863,700 ins. of Bar. Height,

$\nu_2' =$ 6·440, $\nu_2 =$ 6·364,924, $\sigma =$ 0·126,144 in. of Bar. Height,

$\nu_3' =$ 3·664, $\nu_3 = -$ 1·588,538, $\beta_1 =$ 0·009,786,

$\nu_4' =$ 122·516, $\nu_4 =$ 121·384,282, $\beta_2 =$ 2·996,236.

The Skewness is ·050.

The distributions of means and σ^2 of samples from the third population (C) are given, with that population, in Parts IV and V.

Tippett's Random Sample.

Tippett's Table of Random Numbers consists of 10,000 random numbers, the method of using them as an aid to sampling being as follows. The numbers range from 0000 to 9999 and, before using the table, each individual of the population is given one of these numbers. The population therefore must not be greater than 10,000 but generally it will be considerably less and consequently only certain of these numbers will be required. Suppose one category of the sampled population consists of 55 individuals, then these 55 individuals may be given the numbers say 0010—0064, and on traversing the table in any systematic manner (down columns or across rows for example) whenever one of these numbers is met an individual of this category is considered drawn. All categories of the original population are similarly dealt with and the table is traversed until the requisite number of "draws" has been made. For example, if 500 samples of 5 are required, 2500 "draws" must be made and these put into groups of 5 in any systematic manner. It may be, as in the present work, that the number of "draws" required to complete the sampling has not been attained when the table has been completely traversed. In this case, the requisite number of extra "draws" must be obtained, either by giving the individuals of the sampled population a new set of numbers out of the 10,000 and traversing the table as before, or by retaining the numbers already given and re-traversing the table in a different order.

The advantages of Tippett's Random Sample are firstly, that its randomness is reasonably established and consequently any set of samples obtained by it may be considered a random set, secondly that it is very much speedier in application than the ordinary method of drawing tickets or other material from a bag, bowl, or urn.

Its disadvantage is that it necessitates the individuals of the sampled population being drawn one at a time, that is the sampling must be conducted on the basis of an infinite sampled population.

When it is definitely desired to draw the whole sample at a time, i.e. a sample on the basis of a "finite" sampled population, the method of drawing objects from a bag or an urn must be employed. The difficulties of shuffling when the objects are tickets is well known, so in this work samplings made on this basis have been taken both by drawing tickets from a bag and also by drawing more symmetrical and easily shuffled material. This material was, in one case a large number of small commercial coloured beads whose homogeneity of size and shape was rough, and in the other case a small number of coloured marbles which approximated much more to the ideal of complete homogeneity. This ideal of complete homogeneity in the individuals representing the sampled population is well known to be necessary in theory and desirable in practice; but in the past, it has scarcely been recognised how necessary as well as desirable it is in practice, especially when sampling from small populations.

Part II. The Distribution of Means of Samples taken from an "infinite" Population.

(*a*) As mentioned in the introduction the relations known to exist between the moment-coefficients of the Distribution of Means of Samples and the corresponding constants of the Sampled Population indicate that, when N, the number in the sample, is large, the Distribution of Means should be well represented by a Normal Frequency Distribution. When $N = 10$, although a similar result may be expected, it is not so clear that a close approach to normality will be obtained in practice. Consequently it was considered of importance to fit a normal curve to the three experimental distributions (I (*a*), III (*a*) and IV (*a*)) employed in this portion of the work.

The results were as follows:

Distribution I (a).

Bar. Height in ins.	Normal Frequency	Actual Frequency in I (*a*)	Bar. Height in ins.	Normal Frequency	Actual Frequency in I (*a*)
29·275—	0·0730	1	*29·825—*	144·0537	158
29·325—	0·2813	1	*29·875—*	130·4000	145
29·375—	0·9491	0	*29·925—*	103·4294	101
29·425—	2·8079	7	*29·975—*	71·9655	63
29·475—	7·2799	9	*30·025—*	43·8715	37
29·525—	16·6427	18	*30·075—*	23·4632	29
29·575—	32·8911	22	*30·125—*	10·9927	13
29·625—	56·6393	52	*30·175—*	4·5189	3
29·675—	89·2672	83	*30·225—*	1·6269	0
29·725—	118·5678	125	*30·275—*	0·5140	0
29·775—	139·5584	132	*30·325—*	0·1423	1

Testing for Goodness of Fit with 17 Groups, this gave quite a fair fit with $\chi^2 = 20{\cdot}41$ and $P = {\cdot}2031$.

Distribution III (a).

Bar. Height in ins.	Normal Frequency	Actual Frequency in III (*a*)	Bar. Height in ins.	Normal Frequency	Actual Frequency in III (*a*)
29·425—	0·9037	2	*29·825—*	163·2387	146
29·475—	2·5000	3	*29·875—*	151·7139	163
29·525—	7·5084	7	*29·925—*	119·2535	123
29·575—	19·0701	20	*29·975—*	79·2800	85
29·625—	40·9641	44	*30·025—*	44·5745	44
29·675—	74·4157	67	*30·075—*	21·1937	20
29·725—	114·3247	109	*30·125—*	8·5221	8
29·775—	148·5447	158	*30·175—*	2·8981	1

Testing for Goodness of Fit with 15 Groups, this gave an excellent fit with $\chi^2 = 7{\cdot}33$ and $P = {\cdot}9197$.

Distribution IV (a).

Value of Mean	Normal Frequency	Actual Frequency in IV (*a*)	Value of Mean	Normal Frequency	Actual Frequency in IV (*a*)
−2·49—	0·3633	1	*2·01—*	159·0085	157
−1·99—	1·5499	2	*2·51—*	118·6741	121
−1·49—	5·4475	6	*3·01—*	72·9834	78
−0·99—	15·7701	16	*3·51—*	36·9808	33
−0·49—	37·6160	44	*4·01—*	15·4375	15
0·01—	73·9223	53	*4·51—*	5·3100	4
0·51—	119·6895	129	*5·01—*	1·5044	0
1·01—	159·6869	152	*5·51—*	0·3512	1
1·51—	175·5419	188			

Testing for Goodness of Fit with 14 Groups, this gave quite a good fit with $\chi^2 = 11{\cdot}42$ and $P = {\cdot}5758$.

It is apparent from these three values of P that the Distribution of Means of Samples approaches normality in practice very closely indeed, even when N is as low as 10, provided the original population is not of an extremely skew character. In fact, for practical applications, the assumption that the Distribution of Means is a Normal Distribution would be quite justified and would lead to quite satisfactory results, except possibly when N was very small indeed, say 2 or 3, or if the sampled population was extremely skew, N being still rather small.

(*b*) However useful the assumption of normality may be when dealing with the distribution of means of samples in purely practical cases, it is highly desirable that a curve for this distribution should be obtained which shows its skewness, slight as this may perhaps be, and which represents generally the distribution from a theoretical point of view.

To this end it was hoped that, commencing with the various Pearson Frequency Curves, the equations of the corresponding Curves of Means of Samples of N might be obtained directly without the use of the method of moments, but success resulted only in the case when the sampled population was of Type III.

The solution of this is now given, but as Populations (A) and (B) are both represented by Type I curves, the result could not be applied to either of them with the expectation of any reasonable accuracy of representation. In point of fact, as the betas of (A) and (B) are such that the points representing them in the usual diagram, although distinctly within the Type I area, are in the neighbourhood of the Type III line, this application was actually attempted by treating the populations as if they were of Type III. As was to be expected however, the Distributions of Means of Samples so deduced showed no agreement whatever with Distributions I(a) and III(a) or IV(a), and consequently the method is not considered worthy of further consideration here.

The Theoretical Curve of Frequency of the Distribution of Means of Samples of N drawn from an "infinite" sampled population which is represented by a Type III Pearson Curve.

The equation $y = y_0 e^{-\gamma x}\left(1+\frac{x}{a}\right)^{\gamma a}$ of the Type III curve can be written $y = y_0' e^{-px} x^p$, where $p = \gamma a$, by transforming the variate to $x = 1 + \frac{x}{a}$. Here the origin has been changed so that the variate is always positive and the unit in which it is measured is $\frac{1}{a}$ times the original unit.

Now the probability that the values $x_1 x_2 \dots x_N$ of the variate x, forming a sample of N from the above population, lie between x_1 and $x_1 + dx_1$, x_2 and $x_2 + dx_2, \dots$, x_N and $x_N + dx_N$, is

$$\text{Constant } (x_1 x_2 \dots x_N)^p e^{-p(x_1+x_2+\dots+x_N)} dx_1 dx_2 \dots dx_N.$$

Changing the variables to

$$\begin{aligned} X_1 &= x_1 + x_2 + \dots + x_N, \\ X_2 &= x_2, \\ X_3 &= x_3, \\ &\dots\dots\dots \\ X_N &= x_N, \end{aligned}$$

we have $J = 1$, and this expression becomes

$$\text{Constant } (X_1 - X_2 - X_3 - \dots - X_N)^p (X_2 X_3 \dots X_N)^p e^{-pX_1} dX_1 dX_2 \dots dX_N.$$

Putting $C_2 = X_1 - X_3 - \dots - X_N$ and integrating for X_2 between 0 and C_2 the expression becomes

$$\text{Constant } e^{-pX_1} (X_3 X_4 \dots X_N)^p dX_1 dX_3 \dots dX_N \int_0^{C_2} (C_2 - X_2)^p X_2^p dX_2,$$

which, putting $X_2 = C_2 Z_2$, becomes

$$\text{Constant } e^{-pX_1} (X_3 X_4 \dots X_N)^p dX_1 dX_3 \dots dX_N . C_2^{2p+1} . B(p+1, p+1),$$

that is

$$\text{Constant } (X_1 - X_3 - \dots - X_N)^{2p+1} (X_3 X_4 \dots X_N)^p . e^{-pX_1} dX_1 dX_3 \dots dX_N.$$

Thus the result of this integration with regard to X_2 is to leave the form of the expression unaltered except that the power of the difference factor has been raised by $p + 1$.

Therefore repeating the process $N-2$ times to eliminate all the X's except X_1, the expression becomes

$$\text{Constant } X_1^{Np+N-1} e^{-pX_1} dX_1.$$

But if m is the Mean of the Sample, $Nm = X_1$, therefore the probability that the value of the Mean of a Sample will lie between m and $m+dm$ is

$$\text{Constant } m^{N(p+1)-1} e^{-Npm} dm,$$

and the Frequency Curve of Means of Samples is of the form

$$y = y_0 m^{N(p+1)-1} e^{-Npm}.$$

To determine y_0 let n be the total frequency of this Distribution of Means of Samples, that is let n be the number of samples taken.

Then
$$n = \int_0^\infty y\,dm$$

$$= y_0 \int_0^\infty m^{N(p+1)-1} e^{-Npm} dm,$$

and putting $Npm = z$,

$$n = \frac{y_0}{(Np)^{N(p+1)}} \int_0^\infty e^{-z} z^{N(p+1)-1} dz$$

$$= \frac{y_0}{(Np)^{N(p+1)}} \Gamma\{N(p+1)\},$$

therefore
$$y_0 = \frac{n(Np)^{N(p+1)}}{\Gamma\{N(p+1)\}}.$$

Thus the curve of frequency of Means of Samples of N drawn from a sampled population represented by a Type III curve is

$$y = \frac{n(Np)^{N(p+1)}}{\Gamma\{N(p+1)\}} m^{N(p+1)-1} e^{-Npm},$$

where n is the total frequency of the distribution.

We now turn to the constants of this Distribution of Means of Samples.

With the notation employed previously for the constants of the Distribution of Means of Samples of N, and n being the total frequency, we have

$$n \cdot {}_1M_1' = y_0 \int_0^\infty e^{-Npm} m^{N(p+1)} dm$$

$$= y_0 \int_0^\infty \frac{e^{-z} z^{N(p+1)}}{(Np)^{N(p+1)+1}} dz, \text{ where } z = Npm$$

$$= y_0 \frac{\Gamma\{N(p+1)+1\}}{(Np)^{N(p+1)+1}},$$

but
$$y_0 = \frac{n(Np)^{N(p+1)}}{\Gamma\{N(p+1)\}}.$$

Therefore
$${}_1M_1' = \frac{1}{Np} \frac{\Gamma\{N(p+1)+1\}}{\Gamma\{N(p+1)\}} = \frac{p+1}{p}.$$

Thus in the original unit

$$ {}_1M_1 = \frac{(p+1)\,a}{p} = \frac{p+1}{\gamma}, $$

which shows that here, as is necessary, the mean of this distribution is the mean of the original population.

Again
$$ n \,.\, {}_1M_2' = y_0 \int_0^\infty e^{-Npm}\, m^{N(p+1)+1}\, dm $$
$$ = y_0 \int_0^\infty \frac{e^{-z}\, z^{N(p+1)+1}}{(Np)^{N(p+1)+2}}\, dz $$
$$ = y_0 \frac{\Gamma\{N(p+1)+2\}}{(Np)^{N(p+1)+2}}. $$

Therefore
$$ {}_1M_2' = \frac{\{N(p+1)+1\}\{N(p+1)\}}{N^2p^2} $$
$$ = \frac{(p+1)^2}{p^2} + \frac{1}{N}\frac{p+1}{p^2}. $$

Therefore
$$ {}_1M_2 (= \Sigma_1^2) = \frac{1}{N}\frac{p+1}{p^2}, $$

i.e.
$$ {}_1M_2 = \frac{1}{N} \,.\, \frac{p+1}{\gamma^2} \text{ in the original unit} $$
$$ = \frac{1}{N} \text{ in terms of the standard-deviation of original population.} $$

Similarly
$$ {}_1M_3' = \frac{\{N(p+1)+2\}\{N(p+1)+1\}\{N(p+1)\}}{N^3p^3} $$
$$ = \frac{1}{p^3}\left\{(p+1)+\frac{2}{N}\right\}\left\{(p+1)+\frac{1}{N}\right\}\{p+1\}, $$

but
$$ {}_1M_3 = {}_1M_3' - 3 \,.\, {}_1M_1' \,.\, {}_1M_2' + 2\,({}_1M_1')^3. $$

Therefore
$$ {}_1M_3 = \frac{2(p+1)}{N^2p^3}, $$

i.e.
$$ {}_1M_3 = \frac{2(p+1)}{N^2\gamma^3} \text{ in the original unit.} $$

Again, in a like manner,

$$ {}_1M_4' = \frac{\left\{(p+1)+\frac{3}{N}\right\}\left\{(p+1)+\frac{2}{N}\right\}\left\{(p+1)+\frac{1}{N}\right\}\{p+1\}}{N^4p^4}, $$

but
$$ {}_1M_4 = {}_1M_4' - 4 \,.\, {}_1M_1' \,.\, {}_1M_3' + 6\,({}_1M_1')^2 \,.\, {}_1M_2' - 3\,({}_1M_1')^4. $$

Therefore
$$ {}_1M_4 = \frac{1}{p^4}\left\{\frac{3(p+1)^2}{N^2} + \frac{6(p+1)}{N^3}\right\}, $$

i.e.
$$ {}_1M_4 = \frac{1}{\gamma^4}\left\{\frac{3(p+1)^2}{N^2} + \frac{6(p+1)}{N^3}\right\} \text{ in the original unit.} $$

Again $$ {}_1B_1 = \frac{({}_1M_3)^2}{({}_1M_2)^3} = \frac{4}{N(p+1)}, $$

and $$ {}_1B_2 = \frac{{}_1M_4}{({}_1M_2)^2} = 3 + \frac{6}{N(p+1)}; $$

and thus $$ 2\,{}_1B_2 = 6 + 3\,{}_1B_1, $$

showing this curve to be of Type III.

Again, the betas of the sampled Type III population were

$$ \bar{\beta}_1 = \frac{4}{p+1}, \quad \bar{\beta}_2 = 3 + \frac{6}{p+1}, $$

and therefore from this we have

$$ {}_1B_1 = \frac{\bar{\beta}_1}{N}, \quad {}_1B_2 - 3 = \frac{\bar{\beta}_2 - 3}{N}, $$

showing that this curve satisfies all the conditions required of a curve of frequency of means of samples.

Thus, for a sampled population of Type III, we have a complete theoretical representation of any distribution of means of samples taken from it.

(*c*) At present, however, the determination of frequency curves for the distributions of means of samples from populations of types other than Type III is restricted to the intermediary use of the method of moments.

The sampled population and its moments being known, the formulae given in Part I can be used to determine the moments of the distribution of means of samples and hence, when the frequency type is determined, the Pearson Frequency Curve representing this distribution can be obtained. To determine the frequency type speedily, consider a pair of rectangular axes $\beta_1 = 0$, $\beta_2 = 0$ (as in *Tables for Statisticians*, XXXV, p. 66) and represent any distribution by the point (β_1, β_2), whose Cartesian coordinates, with regard to the given axes, are its betas. Now considering the two points $(\bar{\beta}_1, \bar{\beta}_2)$, $({}_1B_1, {}_1B_2)$ representing sampled population and distribution of means of samples respectively we have, from Part I, the following relation between their coordinates:

$$ \frac{\bar{\beta}_2 - 3}{\bar{\beta}_1} = \frac{{}_1B_2 - 3}{{}_1B_1} = \lambda, $$

where λ is independent of N the number in the sample, that is λ is a constant for all distributions of means of samples derived from the original population.

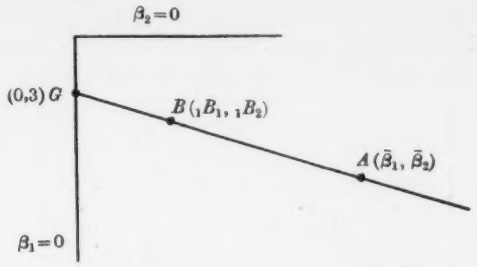
Therefore, given the point A, $(\bar{\beta}_1, \bar{\beta}_2)$, the point B, $({}_1B_1, {}_1B_2)$, lies on the line

$$ {}_1B_2 = \lambda \,.\, {}_1B_1 + 3, $$

which passes through the Gaussian point and the given point A, $(\bar{\beta}_1, \bar{\beta}_2)$.

Now since ${}_1B_1 = \frac{\bar{\beta}_1}{N}$, we have from the diagram $\frac{GB}{GA} = \frac{1}{N}$, and since N is $\geqslant 1$ necessarily, the point B, i.e. $({}_1B_1, {}_1B_2)$, lies on the line considered between G and A, and *Tables for Statisticians*, XXXV shows the frequency type immediately.

It may be noted that, if the original population be any of Types I, II, III, IV or VII and its betas are of moderate size, the frequency type of the distribution of means of samples is the same as that of the sampled population, agreeing with the result obtained in Part II (*b*).



This method was applied to Populations (A) and (B) in order to determine the degree of accuracy with which the actual Distributions I (*a*), III (*a*) and IV (*a*) are represented by the corresponding theoretical curves obtained in this manner.

Population (A) has for its constants:

$$\text{Mean} = 29{\cdot}858{,}0155 \text{ ins. of Bar. Height},$$
$$\bar{\sigma} = 0{\cdot}384{,}1700 \text{ in. of Bar. Height},$$
$$\bar{\beta}_1 = 0{\cdot}189{,}424, \quad \bar{\beta}_2 = 3{\cdot}160{,}978,$$

and its frequency type is Type I. Therefore the Distribution of Means of Samples of 10 is represented by a Type I curve and, from the equations of Part I, its constants are:

$$\text{Mean} = 29{\cdot}858{,}0155 \text{ ins. of Bar. Height},$$
$$\Sigma_1 = 0{\cdot}121{,}4852 \text{ in. of Bar. Height},$$
$${}_1B_1 = 0{\cdot}018{,}942, \quad {}_1B_2 = 3{\cdot}016{,}098.$$

With the usual notation, the fitting of these constants to a Type I curve gives

$$r = 486{\cdot}517{,}905, \quad \epsilon = 37{,}463{\cdot}612{,}112;$$

the quadratic is $m'^2 - 486{\cdot}517{,}905m' + 37{,}463{\cdot}612{,}112 = 0,$

its roots are $390{\cdot}606{,}521$ and $95{\cdot}911{,}385.$

Now as the $\bar{\nu}_2$ of Population (A) is negative, the ${}_1M_3$ of this distribution is also negative and, both the roots of this quadratic being positive, m_1' is the larger root and therefore $m_1 = 389{\cdot}606{,}521$, $m_2 = 94{\cdot}911{,}385$. Using 0·1 in. of Bar. Height as working unit, $b = 67{\cdot}423{,}715$ and $a_1 = 54{\cdot}216{,}202$, $a_2 = 13{\cdot}207{,}422$.

Now
$$y_0 = \frac{N}{b}\,\frac{m_1^{m_1}\,.\,m_2^{m_2}}{(m_1+m_2)^{m_1+m_2}}\,\frac{\Gamma(m_1+m_2+2)}{\Gamma(m_1+1)\,.\,\Gamma(m_2+1)},$$

therefore
$$\log y_0 = 2{\cdot}516{,}5925,$$

and
$$y_0 = 328{\cdot}5432.$$

The frequency curve representing the Distribution of Means of Samples of 10 *from Population* (A) *is thus*

$$y = 328{\cdot}5432\left(1+\frac{x}{54{\cdot}216{,}202}\right)^{389{\cdot}606{,}521}\left(1-\frac{x}{13{\cdot}207{,}422}\right)^{94{\cdot}911{,}385}.$$

The Mean is 29·858,0155 ins. of Bar. Height and

Mean-Mode = − 0·008,4290 in. of Bar. Height.

Therefore the origin, which is the mode, is at 29·866,445 ins. of Bar. Height, and the range is from $x = -54{\cdot}216{,}202$ to $x = 13{\cdot}207{,}422$. Now the actual distributions of means of samples derived from Population (A) range from 29·275 ins. to 30·325 ins., so it is sufficient to plot this curve from $x = -5{\cdot}0$ to $x = 4{\cdot}5$.

The Plotted Ordinates of this Curve, which is shown in Fig. I, are

x	y	x	y	x	y	x	y
−5·0	0·2390	*−1·5*	159·736	*0·125*	326·837	*2·5*	31·2449
−4·5	0·8729	*−1·0*	237·122	*0·25*	321·595	*3·0*	10·2027
−4·0	2·8415	*−0·75*	273·029	*0·50*	301·400	*3·5*	2·5745
−3·5	8·2122	*−0·50*	302·381	*0·75*	270·046	*4·0*	0·4694
−3·0	20·9904	*−0·25*	321·724	*1·0*	231·044	*4·5*	0·0686
−2·5	47·2366	*−0·125*	326·809	*1·5*	146·257		
−2·0	93·0828	*0·00*	328·543	*2·0*	75·4552		

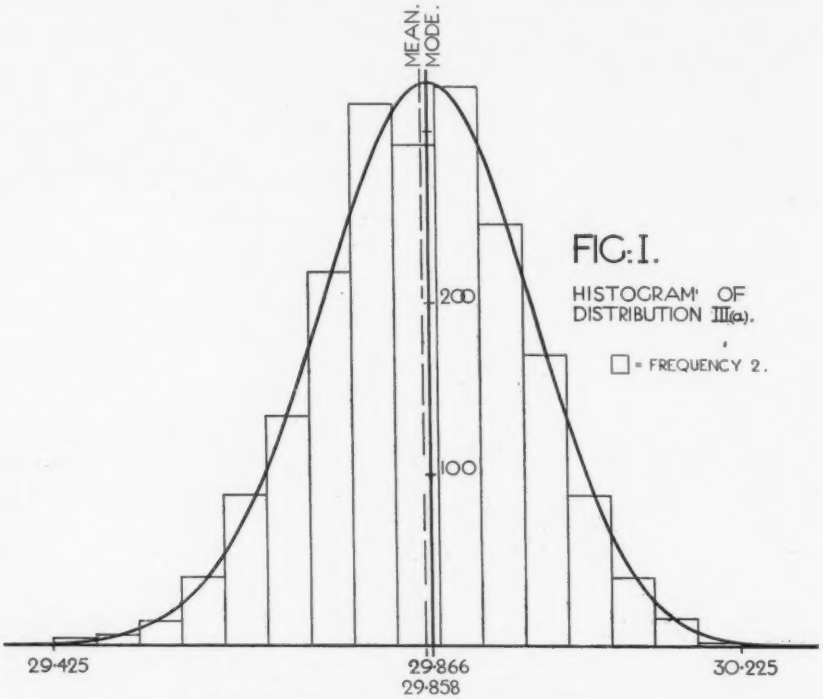


FIG. I.

HISTOGRAM OF DISTRIBUTION III(a).

□ = FREQUENCY 2.

Means of Samples in $\frac{1}{10}$ inch of Barometric Height.

Distribution of Means in 1000 Samples of 10 from the Skew Distribution of Barometric Heights (A). Fitted Curve from theoretical values of σ, β_1 and β_2 for curve of means.

Again, Population (B) has constants

$$\text{Mean} = 1{\cdot}8518, \qquad \bar{\sigma} = 3{\cdot}569{,}207,$$
$$\bar{\beta}_1 = 0{\cdot}219{,}333, \qquad \bar{\beta}_2 = 3{\cdot}157{,}676,$$

and its frequency type is Type I. Therefore the curve representing the distribution of means of samples of 10 is of Type I, with constants

$$\text{Mean} = 1{\cdot}85818, \qquad \Sigma_1 = 1{\cdot}128{,}682,$$
$${}_1B_1 = 0{\cdot}021{,}933, \qquad {}_1B_2 = 3{\cdot}015{,}768.$$

We have, as before,

$$r = 349{\cdot}152{,}438, \qquad \epsilon = 20{,}554{\cdot}430{,}847,$$
$$m_1 = 273{\cdot}187{,}591, \qquad m_2 = 73{\cdot}964{,}847,$$
$$b = 51{\cdot}434{,}693, \qquad a_1 = 40{\cdot}476{,}020,$$
$$a_2 = 10{\cdot}958{,}780, \qquad y_0 = 353{\cdot}5420.$$

The frequency curve representing the Distribution of Means of Samples of 10 *from Population* (B) *is thus*

$$y = 353{\cdot}5420\left(1 + \frac{x}{40{\cdot}476{,}020}\right)^{273{\cdot}187{,}591}\left(1 - \frac{x}{10{\cdot}958{,}780}\right)^{73{\cdot}964{,}847}.$$

The Mean is 1·858,18 and Mean-Mode $= -0{\cdot}084{,}541$. Therefore the origin, which is the mode, is 1·942,721, and the range is from $x = -40{\cdot}476{,}020$ to $x = 10{\cdot}958{,}780$. Now the actual distribution (IV (a)) of means of samples of 10 derived from Population (B) ranges from $-2{\cdot}5$ to 6·0 units, so it is sufficient to plot the curve from $x = -4{\cdot}5$ to $x = 4{\cdot}5$.

The Plotted Ordinates of this Curve, which is shown in Fig. II, are

x	y	x	y	x	y	x	y
$-4{\cdot}5$	0·4137	$-1{\cdot}0$	242·950	$0{\cdot}25$	344·913	$3{\cdot}0$	5·6736
$-4{\cdot}0$	1·5782	$-0{\cdot}75$	285·652	$0{\cdot}50$	319·821	$3{\cdot}5$	1·0692
$-3{\cdot}5$	5·2649	$-0{\cdot}50$	322·369	$0{\cdot}75$	281·500	$4{\cdot}0$	0·1385
$-3{\cdot}0$	15·2917	$-0{\cdot}25$	345·082	$1{\cdot}0$	234·611	$4{\cdot}5$	0·0118
$-2{\cdot}5$	38·4765	$-0{\cdot}125$	351·396	$1{\cdot}5$	137·218		
$-2{\cdot}0$	83·3704	$0{\cdot}00$	353·542	$2{\cdot}0$	62·7644		
$-1{\cdot}5$	154·493	$0{\cdot}125$	351·382	$2{\cdot}5$	21·9485		

Tests for Goodness of Fit.

With the obvious exception of the tests for normality, the data for all tests of Goodness of Fit of the theoretical curves against the corresponding experimental distributions were obtained by first drawing the curves accurately to a large scale on prepared cardboard, and then planimetering the areas corresponding to the frequency groups of the experimental distributions. This is the most satisfactory method in practice, but is somewhat lengthy and laborious.

In this case, the tests for Goodness of Fit gave

(1) For theoretical curve of means of samples of 10 from Population (A) against experimental Distribution I (*a*), i.e. ticket drawing:

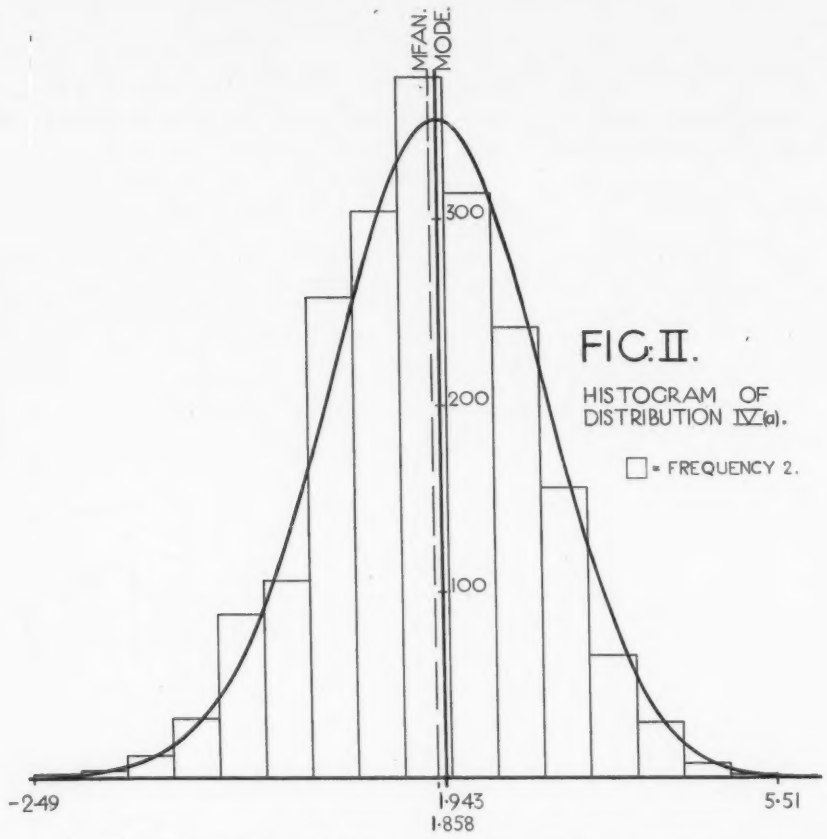
15 Groups. $\chi^2 = 84{\cdot}07$. $P = {\cdot}0000$.

(2) For theoretical curve of means of samples of 10 from Population (A) against experimental Distribution III (*a*):

16 Groups. $\chi^2 = 7{\cdot}01$. $P = {\cdot}9573$.

(3) For theoretical curve of means of samples of 10 from Population (B) against experimental Distribution IV (*a*):

16 Groups. $\chi^2 = 18{\cdot}70$. $P = {\cdot}2284$.



Means of Samples in units of original Distribution (B) with sub-range $\frac{1}{2}$ unit.

Distribution of Means of 1000 Samples of 10 from the Smooth Skew Distribution (B). Fitted Curve from theoretical values of σ, β_1 and β_2 for curve of means.

On the whole these theoretical curves of means of samples do not give as good fits to the actual distributions as did the Normal Curve. There is little in this beyond the fact that for mere purposes of representing distributions of means of small samples, the Normal Curve will give in all probability as good or better results as any other method, and therefore may be preferred often in practice on account of its relative simplicity.

In the two cases when the sampling was carried out by Tippett's Random Sample, that is when the randomness could be considered established, the fits to theory were $P = \cdot 9573$ and $\cdot 2284$; consequently we may conclude, fairly, that the Pearson Frequency Curves employed here as a result of the equations

$$_1B_1 = \frac{\bar{\beta}_1}{N} \text{ and } _1B_2 - 3 = \frac{\bar{\beta}_2 - 3}{N}$$

do represent adequately actual distributions of means of samples provided the assumption of an "infinite" sampled population be made and the randomness of the sampling is reasonably certain.

It is fairly clear from this, that the poor fit of Distribution I (a) must be due to lack of effective shuffling of the tickets, although great care was employed in the attempt to ensure this shuffling was effective and the resulting samples to be of a quite random character. This difficulty is quite well known of course, but can be overcome to a great extent by the use of beads or similar material instead of the tickets. This point is dealt with in Part IV.

PART III. THE DISTRIBUTION OF SECOND MOMENTS (σ^2) OF SMALL SAMPLES FROM ANY "INFINITE" POPULATION.

(a) Commencing with the various Pearson Curves as types of sampled populations, many attempts were made to deduce without the use of moments the corresponding theoretical frequency curves of σ^2 for samples. These were not successful, so recourse was necessarily taken to the method of moments.

In *Biometrika*, Vol. XII. p. 193 (23), p. 194 (24) and (25), Professor Al. Tchouproff has given formulae for the moments of the distribution of σ^2 of samples from any "infinite" population in terms of the moments of that population. In *Biometrika*, Vol. XVII. pp. 79—83, I have obtained these formulae afresh by a shorter method and have corrected an error in the expression for the 4th moment of the distribution of σ^2 of samples. This investigation was carried out, because Professor Tchouproff's results followed as a result of some very intricate algebra, whilst on close inspection the expression for the 4th moment was seen to be in error, so it was considered of some value, not only to correct the expression for the 4th moment, but also to check the others and to deduce all of them by a simpler analysis more in conformity with English statistical methods.

The formulae*, thus corrected, for the moments of the distribution of σ^2 of samples are

$$\text{Mean } (= {}_2M_1') = \bar{\nu}_2\left(1 - \frac{1}{N}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$${}_2M_2 = \frac{1}{N}[\bar{\nu}_4 - \bar{\nu}_2^2] - \frac{2}{N^2}[\bar{\nu}_4 - 2\bar{\nu}_2^2] + \frac{1}{N^3}[\bar{\nu}_4 - 3\bar{\nu}_2^2] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$\begin{aligned}{}_2M_3 = {} & \frac{1}{N^2}[\bar{\nu}_6 - 3\bar{\nu}_4\bar{\nu}_2 - 6\bar{\nu}_3^2 + 2\bar{\nu}_2^3] \\ & - \frac{1}{N^3}[3\bar{\nu}_6 - 21\bar{\nu}_4\bar{\nu}_2 - 18\bar{\nu}_3^2 + 26\bar{\nu}_2^3] + \frac{1}{N^4}[3\bar{\nu}_6 - 33\bar{\nu}_4\bar{\nu}_2 - 22\bar{\nu}_3^2 + 54\bar{\nu}_2^3] \\ & - \frac{1}{N^5}[\bar{\nu}_6 - 15\bar{\nu}_4\bar{\nu}_2 - 10\bar{\nu}_3^2 + 30\bar{\nu}_2^3] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),\end{aligned}$$

$$\begin{aligned}{}_2M_4 = {} & \frac{3}{N^2}[\bar{\nu}_4 - \bar{\nu}_2^2]^2 + \frac{1}{N^3}[\bar{\nu}_8 - 4\bar{\nu}_6\bar{\nu}_2 - 24\bar{\nu}_5\bar{\nu}_3 - 15\bar{\nu}_4^2 + 48\bar{\nu}_4\bar{\nu}_2^2 + 96\bar{\nu}_3^2\bar{\nu}_2 - 30\bar{\nu}_2^4] \\ & - \frac{1}{N^4}[4\bar{\nu}_8 - 40\bar{\nu}_6\bar{\nu}_2 - 96\bar{\nu}_5\bar{\nu}_3 - 54\bar{\nu}_4^2 + 336\bar{\nu}_4\bar{\nu}_2^2 + 528\bar{\nu}_3^2\bar{\nu}_2 - 306\bar{\nu}_2^4] \\ & + \frac{1}{N^5}[6\bar{\nu}_8 - 96\bar{\nu}_6\bar{\nu}_2 - 176\bar{\nu}_5\bar{\nu}_3 - 102\bar{\nu}_4^2 + 924\bar{\nu}_4\bar{\nu}_2^2 + 1232\bar{\nu}_3^2\bar{\nu}_2 - 1044\bar{\nu}_2^4] \\ & - \frac{1}{N^6}[4\bar{\nu}_8 - 88\bar{\nu}_6\bar{\nu}_2 - 160\bar{\nu}_5\bar{\nu}_3 - 95\bar{\nu}_4^2 + 1050\bar{\nu}_4\bar{\nu}_2^2 + 1360\bar{\nu}_3^2\bar{\nu}_2 - 1395\bar{\nu}_2^4] \\ & + \frac{1}{N^7}[\bar{\nu}_8 - 28\bar{\nu}_6\bar{\nu}_2 - 56\bar{\nu}_5\bar{\nu}_3 - 35\bar{\nu}_4^2 + 420\bar{\nu}_4\bar{\nu}_2^2 + 560\bar{\nu}_3^2\bar{\nu}_2 - 630\bar{\nu}_2^4] \\ & \ldots\ldots\ldots\ldots(4),\end{aligned}$$

where ${}_2M_2$, ${}_2M_3$, etc. are the moments of the distribution of σ^2 about its mean ${}_2M_1'$, $\bar{\nu}_2$, $\bar{\nu}_3$, etc. are the uncorrected moments of the sampled population about its mean, and N is the number in the sample.

Tchouproff's formulae may be expressed in terms of the betas of the sampled population as follows:

$$\text{Mean} = \left(1 - \frac{1}{N}\right)\bar{\nu}_2,$$

$${}_2M_2 = \frac{\bar{\nu}_2^2}{N}\left(1 - \frac{1}{N}\right)\left\{\bar{\beta}_2 - 1 - \frac{\bar{\beta}_2 - 3}{N}\right\},$$

$$\begin{aligned}{}_2M_3 = {} & \frac{\bar{\nu}_2^3}{N^2}\left\{\bar{\beta}_4 - 3\bar{\beta}_2 - 6\bar{\beta}_1 + 2 - \frac{1}{N}(3\bar{\beta}_4 - 21\bar{\beta}_2 - 18\bar{\beta}_1 + 26)\right. \\ & \left. + \frac{1}{N^2}(3\bar{\beta}_4 - 33\bar{\beta}_2 - 22\bar{\beta}_1 + 54) - \frac{1}{N^3}(\bar{\beta}_4 - 15\bar{\beta}_2 - 10\bar{\beta}_1 + 30)\right\},\end{aligned}$$

$$\begin{aligned}{}_2M_4 = {} & \frac{\bar{\nu}_2^4}{N^2}\left\{3(\bar{\beta}_2 - 1)^2 + \frac{1}{N}(\beta_6 - 4\bar{\beta}_4 - 24\bar{\beta}_3 - 15\bar{\beta}_2^2 + 48\bar{\beta}_2 + 96\bar{\beta}_1 - 30)\right. \\ & - \frac{1}{N^2}(4\bar{\beta}_6 - 40\bar{\beta}_4 - 96\bar{\beta}_3 - 54\bar{\beta}_2^2 + 336\bar{\beta}_2 + 528\bar{\beta}_1 - 306) \\ & + \frac{1}{N^3}(6\bar{\beta}_6 - 96\bar{\beta}_4 - 176\bar{\beta}_3 - 102\bar{\beta}_2^2 + 924\bar{\beta}_2 + 1232\bar{\beta}_1 - 1044) \\ & - \frac{1}{N^4}(4\bar{\beta}_6 - 88\bar{\beta}_4 - 160\bar{\beta}_3 - 95\bar{\beta}_2^2 + 1050\bar{\beta}_2 + 1360\bar{\beta}_1 - 1395) \\ & \left. + \frac{1}{N^5}(\bar{\beta}_6 - 28\bar{\beta}_4 - 56\bar{\beta}_3 - 35\bar{\beta}_2^2 + 420\bar{\beta}_2 + 560\bar{\beta}_1 - 630)\right\}.\end{aligned}$$

[* The formulae (1) and (2) were known before the publication of Prof. Tchouproff's paper. Ed.]

These formulae, though theoretically exact, suffer from the very considerable disadvantage that they involve the first eight moments of the sampled population. In the first place the large probable errors of the higher moments of the sampled population (this is in general merely a larger sample from a still larger population) cause formulae involving their use to be of doubtful value in practical statistics, and secondly the labour involved in calculating the higher of these eight moments is too great to be undertaken in ordinary statistical practice.

(*b*) The consideration of the second of these points will be left to the next section, the purpose of the present section being to determine whether or no these formulae do give good representations of actual experimental distributions of σ^2 of samples in spite of the fact that they, the formulae, contain high moments whose exact size is questionable. To this end Tchouproff's formulae were applied directly to the betas of Populations (A) and (B) given in the Introduction.

The results obtained were:

(1) *The Distribution of σ^2 of Samples of* 10, *obtained by the application of Tchouproff's Formulae to the actual uncorrected Moments of Population* (A), *had for its constants:*

$$\text{Mean} = 13{\cdot}282{,}791, \qquad {}_2M_2 = 42{\cdot}047{,}408,$$
$${}_2M_3 = 360{\cdot}510{,}852, \qquad {}_2M_4 = 11{,}747{\cdot}805{,}045,$$

and Standard Deviation

$$(\Sigma_2) = 6{\cdot}484{,}397,$$

which gave betas of

$${}_2B_1 = 1{\cdot}748{,}312, \qquad {}_2B_2 = 6{\cdot}644{,}744.$$

(2) *The Distribution of σ^2 of Samples of* 10, *obtained by the application of Tchouproff's Formulae to the actual uncorrected Moments of Population* (B), *had for its constants:*

$$\text{Mean} = 11{\cdot}465{,}313, \qquad {}_2M_2 = 31{\cdot}284{,}573,$$
$${}_2M_3 = 179{\cdot}342{,}927, \qquad {}_2M_4 = 4{,}445{\cdot}996{,}023,$$

and Standard Deviation

$$(\Sigma_2) = 5{\cdot}593{,}261,$$

which gave betas of

$${}_2B_1 = 1{\cdot}050{,}297, \qquad {}_2B_2 = 4{\cdot}541{,}956.$$

Tables of comparison of these constants with the corresponding constants of the actual Distributions I (*b*), III (*b*) and IV (*b*) of σ^2 of 1000 samples of 10 from Populations (A) and (B) are

Table for the Distributions from Population (A).

Constant	Theoretic Value	Value from I (*b*)	Value from III (*b*)
Mean	13·282,791	$15{\cdot}848 \pm 0{\cdot}138$	$13{\cdot}725 \pm 0{\cdot}138$
${}_2M_2$	42·047,408	$61{\cdot}162{,}896 \pm 2{\cdot}130{,}785$	$45{\cdot}829{,}375 \pm 2{\cdot}130{,}785$
${}_2M_3$	360·510,852	$701{\cdot}313{,}792 \pm 74{\cdot}847{,}038$	$505{\cdot}969{,}031 \pm 74{\cdot}847{,}038$
${}_2M_4$	11,747·805,045	$24{,}542{\cdot}441{,}607 \pm 5{,}299{\cdot}341{,}004$	$17{,}895{\cdot}282{,}136 \pm 5{,}299{\cdot}341{,}004$
${}_2B_1$	1·7483	$2{\cdot}1496 \pm 0{\cdot}5379$	$2{\cdot}6596 \pm 0{\cdot}5379$
${}_2B_2$	6·6447	$6{\cdot}5606 \pm 2{\cdot}6844$	$8{\cdot}5202 \pm 2{\cdot}6844$
Σ_2	6·4844	$7{\cdot}8207 \pm 0{\cdot}0978$	$6{\cdot}7697 \pm 0{\cdot}0978$

Table for the Distributions from Population (B).

Constant	Theoretic Value	Value from IV (*b*)
Mean	11·465,313	11·609 ± 0·119
$_2M_2$	31·284,573	32·204,119 ± 1·255,819
$_2M_3$	179·342,927	194·848,209 ± 22·569,784
$_2M_4$	4,445·996,023	4,857·431,837 ± 702·936,762
$_2B_1$	1·0503	1·1367 ± 0·1788
$_2B_2$	4·5420	4·6836 ± 0·5087
Σ_2	5·5933	5·6749 ± 0·0843

In these tables the P.E.'s of the constants of the experimental distributions were obtained (in accordance with theory) from the constants of the corresponding theoretic distributions. The higher betas $_2B_3$, $_2B_5$ and $_2B_6$ of the two theoretic distributions necessary to calculate the P.E.'s of $_2M_3$ and $_2M_4$ of the experimental distributions were calculated from the $_2B_1$ and $_2B_2$ of the theoretic distributions by use of the difference formulae of the next section, this being the only feasible method. This fact, together with the general theory upon which the P.E. rests, causes the values of the P.E.'s given here to be of doubtful accuracy and utility. The indication they give of the size of the variations of the constants due to sampling is probably only of a quite approximate nature.

The tables show very little beyond the large variations to be expected in the betas and higher moments of actual distributions of σ^2 of samples. The subsequent tests for Goodness of Fit show that these variations have little significance; for example, although the agreement between the constants of Distribution III (*b*) and the theoretic constants in the first table is strikingly less close than that between the constants of Distribution IV (*b*) and the theoretic constants in the second table, the fit in the first case is rather better than that in the second.

With the application of the Goodness of Fit tests in view these two theoretic distributions of σ^2 of samples of 10 from Populations (A) and (B) were now fitted with Pearson Frequency Curves.

The Constants of the first being

$$\text{Mean} = 13{\cdot}282{,}791, \qquad {}_2B_1 = 1{\cdot}748{,}312,$$
$$_2B_2 = 6{\cdot}644{,}744, \qquad \Sigma_2 = 6{\cdot}484{,}397,$$

the Skewness is ·932 and it is represented by a Type IV curve with equation of the form

$$y = y_0 \left(1 + \frac{x^2}{a^2}\right)^{-m} e^{-\nu \tan^{-1}\frac{x}{a}},$$

where, in the usual notation,

$$r = 11{\cdot}434{,}579, \qquad m = 6{\cdot}717{,}290,$$
$$z = 1{,}926{\cdot}017{,}102, \qquad \nu = -42{\cdot}370{,}612, \qquad a = 5{\cdot}457{,}542.$$

Taking $$y_0 = \frac{N}{a}\sqrt{\frac{r}{2\pi}}\cdot\frac{e^{\frac{\cos^2\phi}{r}-\frac{1}{12r}-\phi\nu}}{(\cos\phi)^{r+1}},$$

where $$\tan\phi = \frac{\nu}{r} = -3{\cdot}705{,}4807,$$

we have $$\log y_0 = -14{\cdot}416{,}779,$$

and the curve is therefore

$$y = 10^{-14{\cdot}416{,}779}\left[1+\frac{x^2}{(5{\cdot}457{,}542)^2}\right]^{-6{\cdot}717{,}290}\; e^{42{\cdot}370{,}612\tan^{-1}\left(\frac{x}{5{\cdot}457{,}542}\right)}.$$

The Origin $$= \text{mean} + \frac{\nu a}{r} = -6{\cdot}940{,}026.$$

The Mode $$= \text{mean} - \frac{1}{2}\frac{{}_2M_3}{{}_2M_2}\frac{r-2}{r+2} = 10{\cdot}272{,}229,$$

which corresponds to $\theta = 72^\circ\ 25{\cdot}6'$.

Putting $x = a\tan\theta$ the curve becomes

$$\log y = -14{\cdot}416{,}779 + 13{\cdot}434{,}579\log\cos\theta + 18{\cdot}401{,}324\,\theta.$$

Again, Distributions I (b) and III (b) range from $\nu_2 = 0{\cdot}5$ to $\nu_2 = 55{\cdot}5$, so it is sufficient to plot from $\theta = 53^\circ$ to $\theta = 84^\circ$.

The Plotted Ordinates of this Curve, which is shown in Fig. III, are

θ°	x	y	θ°	x	y	θ°	x	y
53	7·2424	0·4386	*68*	13·5079	49·3761	*74·5*	19·6793	64·6459
55	7·7942	1·0093	*68·5*	13·8548	53·2682	*75*	20·3678	60·8621
57	8·4039	2·2095	*69*	14·2174	57·0318	*75·5*	21·1027	56·4311
59	9·0620	4·5768	*69·5*	14·5969	60·5757	*76*	21·8890	51·4749
61	9·8457	8·9122	*70*	14·9945	63·8058	*76·5*	22·7323	46·1399
62	10·2642	12·1219	*70·5*	15·4116	66·6226	*77*	23·6392	40·5882
63	10·7110	16·1846	*71*	15·8498	68·9275	*78*	25·6757	29·5106
64	11·1896	21·1833	*71·5*	16·3109	70·6270	*79*	28·0766	19·5119
65	11·7037	27·1414	*72*	16·7966	71·6367	*80*	30·9513	11·5241
66	12·2578	33·9870	*72·5*	17·3091	71·8859	*81*	34·4576	5·9385
66·5	12·5515	37·6856	*73*	17·8508	71·3242	*82*	38·8324	2·5862
67	12·8572	41·5184	*73·5*	18·4243	69·9243	*83*	44·4481	0·9103
67·5	13·1757	45·4356	*74*	19·0327	67·6879	*84*	62·3800	0·0442

The Constants of the second of these theoretic distributions of σ^2 of samples of 10 being

$$\text{Mean} = 11{\cdot}465{,}313, \qquad {}_2B_1 = 1{\cdot}050{,}297,$$
$${}_2B_2 = 4{\cdot}541{,}956, \qquad \Sigma_2 = 5{\cdot}593{,}261,$$

its Skewness is ·522 and it is represented by a Type III curve with equation of the form

$$y = y_0 e^{-\gamma x}\left(1+\frac{x}{a}\right)^p, \text{ where } p = \gamma a.$$

In the usual notation we have for this curve

$$\gamma = 0{\cdot}348{,}880, \qquad p = 2{\cdot}808{,}447,$$
$$a = 8{\cdot}049{,}894.$$

Also $$y_0 = \frac{N}{a}\frac{p^{p+1}}{e^p\,\Gamma(p+1)}, \text{ i.e. } y_0 = 80{\cdot}6342.$$

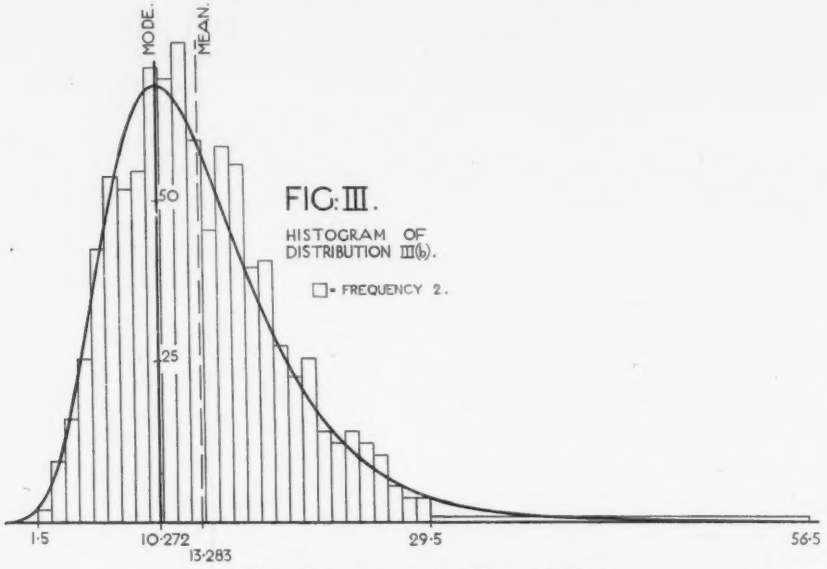
The equation of the curve is therefore

$$y = 80{\cdot}6342\, e^{-0{\cdot}348,880\,x}\left(1+\frac{x}{8{\cdot}049,894}\right)^{2{\cdot}808,447}.$$

The Mean of the distribution is 11·465,313.

The Origin (which is the Mode) = 8·598,998.

The actual Distribution IV (*b*) ranges from $\nu_2 = 0{\cdot}5$ to $\nu_2 = 40{\cdot}5$, so it is sufficient to plot from $x = -8{\cdot}0$ to $x = 32{\cdot}0$.



Distribution of σ^2 in 1000 Samples of 10 from the Skew Distribution of Barometric Heights (A). Fitted Curve from actual moments of (A).

The Plotted Ordinates of this Curve, which is shown in Fig. IV, are

x	y	x	y	x	y	x	y
-8	0·0008	$-0{\cdot}5$	80·1800	*8*	34·3576	*24*	0·9022
-7	3·0386	*0·0*	80·6342	*10*	23·7808	*26*	0·5322
-6	14·0355	*0·5*	80·2157	*12*	15·8987	*28*	0·3109
-5	30·2227	*1·0*	79·0353	*14*	10·3348	*30*	0·1801
-4	47·2824	*2*	74·8409	*16*	6·5639	*32*	0·1035
-3	61·9925	*3*	68·9160	*18*	4·0885		
-2	72·6420	*4*	62·0113	*20*	2·5047		
-1	78·7497	*6*	47·5042	*22*	1·5126		

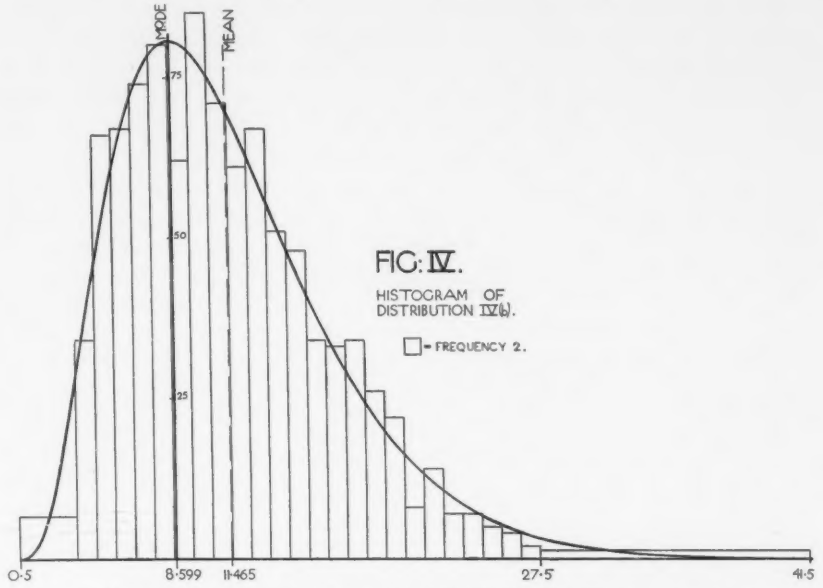
Tests for Goodness of Fit.

(1) For theoretical curve of σ^2 of samples of 10 from Population (A) against experimental Distribution I (*b*), i.e. ticket drawing:

29 Groups. $\chi^2 = 169{\cdot}95$. $P = 0$.

(2) For theoretical curve of σ^2 of samples of 10 from Population (A) against experimental Distribution III (*b*):

29 Groups. $\chi^2 = 23{\cdot}97$. $P = {\cdot}6831$.



σ^2's of Samples in squared units of original Distribution (B) with sub-range one unit.

Distribution of σ^2 in 1000 Samples of 10 from the Smooth Skew Distribution (B).
Fitted Curve from actual moments of (B).

(3) For theoretical curve of σ^2 of samples of 10 from Population (B) against experimental Distribution IV (*b*):

26 Groups. $\chi^2 = 22{\cdot}46$. $P = {\cdot}6090$.

From these it may be concluded that the theoretical Pearson Curve of σ^2 of samples of N from any "infinite" population, as obtained from its moments given by the Tchouproff formulae (1), (2), (3) and (4), does give an adequate representation of actual distributions for cases where N is small, provided the sampling leading to the actual distribution is of a reasonably random character. Moreover there is nothing to suggest that this representation will be any less accurate for larger values of N, always providing the assumption of an "infinite" sampled population be legitimate.

Population (A) is a sampled population with frequency groups of irregular size and with a very pronounced "tail," whereas Population (B) is one with graduated frequency groups and no pronounced "tail." From the fits obtained here it is evident that Tchouproff's formulae give as good results in one case as the other, that is: the presence of a "tail" has little significance for the purpose of sampling provided the sampling is random. It is very interesting, however, to see the divergences its presence causes between the actual and theoretical values of the higher moments of the distribution of the σ^2 of samples and still further the negligible ultimate effect that arises from these divergences as measured by χ^2.

(*c*) From the previous section Tchouproff's formulae may be deemed to give an efficient representation in practice of actual distributions of σ^2 of samples from any "infinite" population, and therefore it is of importance to ascertain whether their practical use can be extended by eliminating the laborious process of determining the first eight moments of the sampled population.

Now the sampled population being in general merely a sample from a larger population, the actual values of its higher moments are problematical as they are subject to large Probable Errors. Since Tchouproff's formulae can be expressed in terms of the betas and second moment of the sampled population, it is possible therefore that quite good approximate results for the moments of the distribution of σ^2 of samples may be obtained if, instead of the actual higher betas of the sampled population, any convenient good approximations to them are used. The convenience here will consist mainly in the ease and speed of the calculation necessary to obtain these approximate values.

Now assuming the applicability of Pearson's Frequency Curves, the values of $\bar{\beta}_3$, $\bar{\beta}_4$ and $\bar{\beta}_6$ can be obtained theoretically from the actual values of $\bar{\beta}_1$ and $\bar{\beta}_2$ by the use of the difference formulae

$$\beta_n\,(\text{even}) = (n+1)\{\tfrac{1}{2}\beta_{n-1} \quad + (1+\tfrac{1}{2}\alpha)\,\beta_{n-2}\}/(1-\tfrac{1}{2}\overline{n-1}\alpha),$$

$$\beta_n\,(\text{odd}) \;= (n+1)\{\tfrac{1}{2}\beta_1\beta_{n-1} + (1+\tfrac{1}{2}\alpha)\,\beta_{n-2}\}/(1-\tfrac{1}{2}\overline{n-1}\alpha),$$

where $$\alpha = (2\beta_2 - 3\beta_1 - 6)/(\beta_2 + 3),$$

and the purpose of this section is to determine whether the representation of actual distributions of σ^2 of samples is as good, when in Tchouproff's formulae the actual $\bar{\beta}_3$, $\bar{\beta}_4$ and $\bar{\beta}_6$ of the sampled population are replaced by their approximate values obtained by the use of the above difference formulae, as it was when these higher betas were given their actual values as in the previous section.

Before the actual work performed with this object is given, it may be noted that the Tables XLII (*a*), (*b*), (*c*) and (*d*) of the higher betas calculated from β_1 and β_2 by these difference formulae, given in *Tables for Statisticians*, pp. 78—79, are in one or two cases in error. They have been re-calculated lately by Professor K. Yasukawa and are re-issued in the present number of *Biometrika*. Also, what is of more importance here, these difference formulae are based on the assumption that the sampled population is represented by a smooth curve, that is it is a continuous distribution. Therefore to apply them properly the $\bar{\nu}_2$, $\bar{\nu}_3$, $\bar{\nu}_4$ of the

sampled population should be first corrected by Sheppard's corrections and the $\bar{\beta}_1$ and $\bar{\beta}_2$ calculated from these values, then the difference formulae (or tables) should be applied to obtain the higher betas of the sampled population and the Tchouproff formulae applied to them to obtain the first four moments of the distribution of σ^2 of samples of N. Sheppard's corrections reversed should then be applied to these four moments and from the resulting uncorrected moments the two working values of the betas of this distribution obtained.

This procedure is such that, if carried out, the time and labour involved would be little less, if any, than that involved in the calculation of the first eight moments of the sampled population. Consequently, unless the straightforward application of these difference formulae as they stand gives a reliable representation of actual distributions of σ^2 of samples there is, for practical statistics, little point in their use.

With this consideration in view, in the following work, the difference formulae have been applied quite straightforwardly to the first two betas of the sampled populations to obtain the values of the higher betas to be substituted in the Tchouproff formulae.

The Constants of the original Populations (A) and (B), when the higher betas were calculated by the difference formulae, were

(1) For Population (A):

$$\bar{\nu}_2 = 14{\cdot}758{,}657, \quad \bar{\beta}_1 = 0{\cdot}189{,}424,$$
$$\bar{\beta}_2 = 3{\cdot}160{,}978, \quad \bar{\beta}_3 = 1{\cdot}865{,}497,$$
$$\bar{\beta}_4 = 19{\cdot}012{,}510, \quad \bar{\beta}_6 = 182{\cdot}728{,}377.$$

(2) For Population (B):

$$\bar{\nu}_2 = 12{\cdot}739{,}237, \quad \bar{\beta}_1 = 0{\cdot}219{,}333,$$
$$\bar{\beta}_2 = 3{\cdot}157{,}676, \quad \bar{\beta}_3 = 2{\cdot}120{,}111,$$
$$\bar{\beta}_4 = 19{\cdot}058{,}578, \quad \bar{\beta}_6 = 182{\cdot}723{,}685.$$

On application of Tchouproff's formulae it was found that the Constants of the Distribution of σ^2 of Samples of 10 from Population (A) were

$$\text{Mean} = 13{\cdot}282{,}791, \quad {}_2M_2 = 42{\cdot}047{,}408,$$
$${}_2M_3 = 292{\cdot}350{,}383, \quad {}_2M_4 = 8{,}615{\cdot}709{,}286,$$

which gave $\Sigma_2 = 6{\cdot}484{,}397$, ${}_2B_1 = 1{\cdot}149{,}714$,

and ${}_2B_2 = 4{\cdot}873{,}181$;

whilst the Constants of the Distribution of σ^2 of Samples of 10 from Population (B) were

$$\text{Mean} = 11{\cdot}465{,}313, \quad {}_2M_2 = 31{\cdot}284{,}573,$$
$${}_2M_3 = 186{\cdot}066{,}245, \quad {}_2M_4 = 4{,}709{\cdot}529{,}436,$$

which gave $\Sigma_2 = 5{\cdot}593{,}261$, ${}_2B_1 = 1{\cdot}130{,}522$,

and ${}_2B_2 = 4{\cdot}811{,}178$.

The fitting of Pearson Curves to these two distributions gave the following results:

For the first, $\kappa = 3{\cdot}736$; consequently a Type VI curve gives the best fit.

The equation is of the form $y = y_0 (x-a)^{q_2}/x^{q_1}$, and in this case

$$r = -54{\cdot}978{,}810, \qquad \epsilon = -276{\cdot}156{,}622;$$

the quadratic is $z^2 + 54{\cdot}978{,}810z - 276{\cdot}156{,}622 = 0$;

its roots are $-59{\cdot}611{,}422$ and $4{\cdot}632{,}612$.

Therefore $q_1 = 60{\cdot}611{,}422$, $q_2 = 3{\cdot}632{,}612$ ($_2M_3$ is positive);

also $$_2M_2 = \frac{a^2\epsilon}{r^2(r+1)}.$$

Accordingly $a = 157{\cdot}615{,}481$;

also $\log y_0 = 133{\cdot}033{,}978$.

Thus we have the curve:

$$y = 10^{133{\cdot}033{,}978}(x - 157{\cdot}615{,}481)^{3{\cdot}632{,}612}\, x^{-60{\cdot}611{,}422}.$$

The Origin $= -157{\cdot}613{,}650$.

The Mode $= \quad 10{\cdot}050{,}406$.

The actual Distributions I (b) and III (b) range from $\nu_2 = 0{\cdot}5$ to $\nu_2 = 55{\cdot}5$, so it is sufficient to plot the curve from $x = 158{\cdot}0$ to $x = 213{\cdot}0$.

The Plotted Ordinates of this Curve, which is shown in Fig. V, are

x	y	x	y	x	y	x	y
158	0·0183	*167*	69·8821	*175*	38·4848	*188*	3·8014
159	1·3106	*167·66*	70·4309	*176*	33·3829	*190*	2·5232
160	6·4578	*168*	70·2998	*177*	28·7042	*193*	1·3469
161	15·7972	*169*	68·5138	*178*	24·4908	*196*	0·7107
162	27·7962	*170*	65·0586	*179*	20·7534	*199*	0·3720
163	40·3752	*171*	60·4533	*180*	17·4800	*202*	0·1937
164	51·7445	*172*	55·1579	*182*	12·2091	*205*	0·1005
165	60·7301	*173*	49·5505	*184*	8·3823	*208*	0·0521
166	66·7905	*174*	43·9209	*186*	5·6762	*213*	0·0174

For the second, $\kappa = 4{\cdot}713$, so these data are likewise best fitted by a Type VI curve.

Here $r = -69{\cdot}690{,}784$, $\epsilon = -326{\cdot}992{,}335$;

the quadratic is $z^2 + 69{\cdot}690{,}784z - 326{\cdot}992{,}335 = 0$;

its roots are $-74{\cdot}103{,}432$ and $4{\cdot}412{,}648$.

Therefore $q_1 = 75{\cdot}103{,}432$ and $q_2 = 3{\cdot}412{,}648$ ($_2M_3$ is positive);

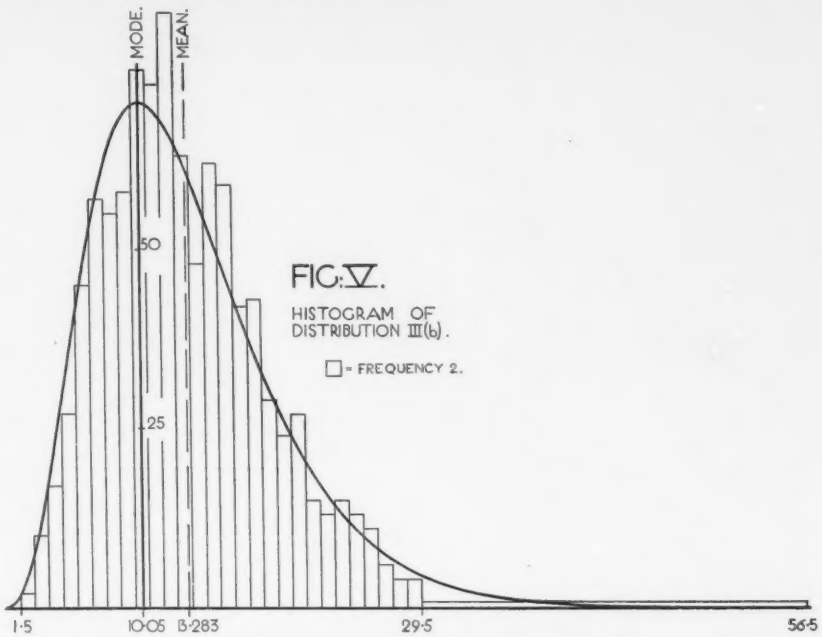
also $a = 178{\cdot}657{,}362$ and $\log y_0 = 169{\cdot}391{,}180$,

and the curve is $$y = 10^{169{\cdot}391{,}180}(x - 178{\cdot}657{,}362)^{3{\cdot}412{,}648}\, x^{-75{\cdot}103{,}432}.$$

The Origin $= -178{\cdot}504{,}191$.

The Mode $= \quad 8{\cdot}657{,}465$.

The actual Distribution IV (*b*) ranges from $\nu_2 = 0{\cdot}5$ to $\nu_2 = 40{\cdot}5$, so it is sufficient to plot the curve from $x = 179{\cdot}0$ to $x = 219{\cdot}0$.



σ^2's of Samples in $\frac{1}{1000}$ sq. inch of Barometric Height.

Distribution of σ^2 in 1000 Samples of 10 from the Skew Distribution of Barometric Heights (A). Fitted Curve from β_1 and β_2 of Distribution (A), higher β's by the Difference Formulae.

The Plotted Ordinates of this Curve, which is shown in Fig. VI, are

x	y	x	y	x	y	x	y
179	0·0404	*187*	81·6889	*194*	41·3504	*206*	3·2752
180	2·8129	*187·16*	81·7382	*195*	34·8630	*208*	2·0171
181	12·4006	*188*	80·5368	*196*	29·0766	*210*	1·2312
182	27·5800	*189*	76·5008	*197*	24·0240	*212*	0·7462
183	44·6443	*190*	70·5232	*198*	19·6863	*214*	0·4497
184	60·1402	*191*	63·4358	*200*	12·9475	*216*	0·2698
185	71·8902	*192*	55·9085	*202*	8·3252	*219*	0·1247
186	79·0347	*193*	48·4341	*204*	5·2587		

The tests for Goodness of Fit gave the following results:

(1) For theoretical curve of σ^2 of samples of 10 from Population (A), when the higher betas of this population have been replaced by the values obtained by the difference formulae, against experimental Distribution I (*b*), i.e. ticket drawing:

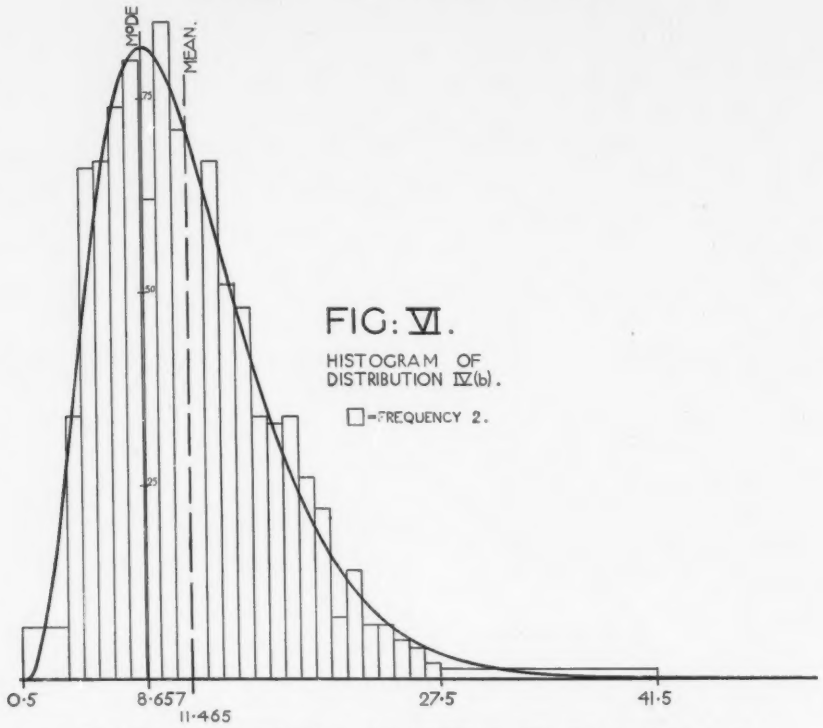
29 Groups. $\chi^2 = 166{\cdot}72$. $P = 0$.

(2) For theoretical curve of σ^2 of samples of 10 from Population (A), when the higher betas of this population have been replaced by the values obtained by the difference formulae, against experimental Distribution III (*b*):

29 Groups. $\chi^2 = 25{\cdot}78$. $P = {\cdot}5851$.

(3) For theoretical curve of σ^2 of samples of 10 from Population (B), when the higher betas of this population have been replaced by the values obtained by the difference formulae, against experimental Distribution IV (*b*):

26 Groups. $\chi^2 = 23{\cdot}99$. $P = {\cdot}5200$.



σ^2's of Samples in squared units of original Distribution (B) with sub-range one unit.

Distribution of σ^2 in 1000 Samples of 10 from the Smooth Skew Distribution (B).
Fitted Curve from β_1 and β_2 of Distribution (B), higher β's by the Difference Formulae.

The fits here are, for all intents and purposes, identical with those obtained in the previous section, consequently the chief difficulty in the application of Tchouproff's formulae is abolished.

We thus conclude that the moments of the distribution of σ^2 of samples of N from any "infinite" population may be deduced with reasonable speed and ease by the application of Tchouproff's formulae for the actual second moment, and

by using the first two betas of the sampled population and the approximate values of its higher betas as obtained by the assumption of Pearson's Frequency Types, together with the straightforward application of the difference relations which follows that assumption. Moreover the values of the moments of the distribution of σ^2 of samples so obtained will lead to a representation of this distribution by a Pearson Frequency Curve, which will describe adequately any actual distributions of σ^2 of samples obtained from an actual set of samples that are the result of truly random sampling.

PART IV. THE DISTRIBUTION OF MEANS OF SAMPLES OF N FROM A FINITE POPULATION OF M.

If, in drawing a sample of N, the individuals are drawn separately and replaced each time before the next is drawn, the sampled population, as we have seen before, can be treated as infinite in all its categories. If however the whole sample be drawn before replacement, the sampled population, in theory, must be treated as of finite total frequency M. However, as we shall see later, even when the samples are actually so drawn in practice, the resulting distributions can be represented approximately under certain circumstances by theoretical curves obtained by treating the sampled population as infinite, the loss in accuracy being small whilst the saving of time and labour is considerable. The determination of reliable conditions under which this can be done is a matter of some difficulty and will be discussed more fully later; but, in order that this consideration should not affect the discussion, in Parts IV and V a sampled population has been chosen which clearly cannot be treated as infinite when the complete sample is drawn before replacement.

This population, which will be called Population (C), is the binomial distribution $(\frac{1}{6}+\frac{5}{6})^2$, having a total frequency of 36.

The distribution, its moments and betas are as follows:

Designation of the Frequency Group	A	B	C
Central Abscissa	0	1	2
Frequency	1	10	25

$$\text{Mean} = 1{\cdot}666{,}667, \qquad \nu_2 = 0{\cdot}277{,}778,$$
$$\nu_3 = -0{\cdot}185{,}185, \qquad \nu_4 = 0{\cdot}277{,}778,$$
$$\nu_5 = -0{\cdot}390{,}950, \qquad \nu_6 = 0{\cdot}620{,}722,$$
$$\nu_8 = 1{\cdot}664{,}817, \qquad \sigma = 0{\cdot}527{,}046,$$
$$\beta_1 = 1{\cdot}600{,}000, \qquad \beta_2 = 3{\cdot}600{,}000,$$
$$\beta_3 = 12{\cdot}160{,}074, \qquad \beta_4 = 28{\cdot}959{,}690$$

and
$$\beta_6 = 279{\cdot}624{,}976.$$

Now the various types of samples of 10 from this population of 36, with their frequencies and means, are:

Type of Sample	Frequency	Mean
0 *A* 0 *B* 10 *C*	3,268,760	2·0
0 *A* 1 *B* 9 *C*	20,429,750	1·9
0 *A* 2 *B* 8 *C*	48,670,875	1·8
0 *A* 3 *B* 7 *C*	57,684,000	1·7
0 *A* 4 *B* 6 *C*	37,191,000	1·6
0 *A* 5 *B* 5 *C*	13,388,760	1·5
0 *A* 6 *B* 4 *C*	2,656,500	1·4
0 *A* 7 *B* 3 *C*	276,000	1·3
0 *A* 8 *B* 2 *C*	13,500	1·2
0 *A* 9 *B* 1 *C*	250	1·1
0 *A* 10 *B* 0 *C*	1	1·0
1 *A* 0 *B* 9 *C*	2,042,975	1·8
1 *A* 1 *B* 8 *C*	10,815,750	1·7
1 *A* 2 *B* 7 *C*	21,631,500	1·6
1 *A* 3 *B* 6 *C*	21,252,000	1·5
1 *A* 4 *B* 5 *C*	11,157,300	1·4
1 *A* 5 *B* 4 *C*	3,187,800	1·3
1 *A* 6 *B* 3 *C*	483,000	1·2
1 *A* 7 *B* 2 *C*	36,000	1·1
1 *A* 8 *B* 1 *C*	1,125	1·0
1 *A* 9 *B* 0 *C*	10	0·9

The total frequency of the samples is, of course, $36!/(10!\,26!) = 254{,}186{,}856$.

Taking the means of samples, we have the following discrete frequency distribution, which is the true or complete distribution of means of samples in this case.

The Complete Distribution of Means of Samples of 10 *from* (C).

Value of the Mean	Frequency
0·9	10
1·0	1,126
1·1	36,250
1·2	496,500
1·3	3,463,800
1·4	13,813,800
1·5	34,640,760
1·6	58,822,500
1·7	68,499,750
1·8	50,713,850
1·9	20,429,750
2·0	3,268,760
Total	254,186,856

The constants are:

$$\text{Mean} = 1{\cdot}666{,}667, \qquad {}_1M_2 = 0{\cdot}020{,}635,$$
$${}_1M_3 = -0{\cdot}000{,}647, \qquad {}_1M_4 = 0{\cdot}001{,}199,$$
$$\Sigma_1 = 0{\cdot}143{,}649, \qquad {}_1B_1 = 0{\cdot}047{,}697,$$
$${}_1B_2 = 2{\cdot}816{,}989.$$

Now the formulae for the Mean, Second Moment-Coefficient and first two Betas of the Distribution of Means of Samples from a finite population are as given below, but their ultimate origin is doubtful, for I have been unable to ascertain the person who first deduced them.

The formulae are:

Mean of Distribution of Means of Samples = Mean of Sampled Population...(1),

$$ {}_1M_2 = \frac{M-N}{N(M-1)}\bar{\nu}_2 \quad \text{.....(2),} $$

$$ {}_1B_1 = \frac{(M-1)(M-2N)^2}{N(M-N)(M-2)^2}\bar{\beta}_1 \quad \text{.....(3),} $$

$$ {}_1B_2 = \frac{(M-1)[(M^2-6MN+M+6N^2)\bar{\beta}_2+3M(M-N-1)(N-1)]}{N(M-2)(M-3)(M-N)} \quad \text{...(4),} $$

where $\bar{\nu}_2$, $\bar{\beta}_1$ and $\bar{\beta}_2$ are the corresponding constants of the sampled population.

On application of these formulae to the constants of Population (C) the results were:

Mean of Distribution of Means = 1·666,667,
$_1M_2$ = 0·020,635,
$_1B_1$ = 0·047,697,
$_1B_2$ = 2·816,989,

which agree with the values given above, thus supplying a numerical check to the formulae.

In the consideration of the Distribution of Means from any population when the complete sample is drawn before replacement, as the above formulae give the betas of the distribution, three considerations arise:

(*a*) The adequate representation of this complete distribution by a Pearson Curve, the constants of which are those given by formulae (1), (2), (3) and (4) of this part.

(*b*) The possible adequate representation of this distribution by a Pearson Curve, the constants of which are the values of $_1M_2$, $_1B_1$ and $_1B_2$ given by the simpler formulae of Part I, which involve the assumption of an infinite sampled population.

(*c*) A Distribution of Means will not be obtained actually in practice by theoretical methods as employed here to obtain the complete distribution from Population (C), but by the actual drawing of a limited number of samples. The question thus arises as to whether the Pearson Curve, as obtained in the first of these considerations, will represent adequately such a distribution of Means obtained by actual sampling.

(*a*) Now it does not follow *a priori* that, because the constants obtained by the formulae of this part are the actual constants of the complete distribution of Means of Samples obtained from the population, the resulting Pearson Curve will represent the distribution, for it is possible for different distributions to have the same second moment and first two betas.

As we shall see in the case of the Distribution of σ^2 it is possible, in extreme cases, for the representation by a Pearson Curve to fail completely. Now, undoubtedly, Population (C) is extreme both in the paucity of its categories and in its smallness, whilst its skewness and lack of a true mode are abnormal. Therefore, if the true distribution of Means of Samples be adequately represented by a Pearson Curve in this case, we are practically justified in assuming that these formulae and a consequent Pearson Curve will represent, for practical purposes, any Distribution of Means of Samples from a finite population.

A Pearson Curve is thus required whose constants are

$$\text{Mean} = 1{\cdot}666{,}667, \qquad \Sigma_1 = 0{\cdot}143{,}649,$$
$$_1B_1 = 0{\cdot}047{,}697, \qquad _1B_2 = 2{\cdot}816{,}989,$$

the unit being that of Population (C).

The curve is clearly of Type I and therefore, in the usual notation, we have

$$r = 20{\cdot}850{,}646, \qquad \epsilon = 101{\cdot}459{,}700,$$
$$m'^2 - 20{\cdot}850{,}646m' + 101{\cdot}459{,}700 = 0,$$

with roots 13·113,664 and 7·736,982; as $_1M_3$ is negative, therefore

$$m_1' = 13{\cdot}113{,}664 \text{ and } m_2' = 7{\cdot}736{,}982;$$

therefore

$$m_1 = 12{\cdot}113{,}664 \text{ and } m_2 = 6{\cdot}736{,}982.$$

Changing, for convenience, the unit to $\frac{1}{100}$ of that of (C), we have

$$b = 13{\cdot}899{,}77, \quad a_1 = 8{\cdot}932{,}13 \text{ and } a_2 = 4{\cdot}967{,}58.$$

Now as the curve is to be used again later and, moreover, the total frequency 254,186,856 of the complete Distribution of Means of Samples of 10 from (C) is so large, the total frequency of the curve has been taken as 1000.

Accordingly: $y_0 = 269{\cdot}791.$

The curve is thus

$$y = 269{\cdot}791\left(1 + \frac{x}{8{\cdot}932{,}13}\right)^{12{\cdot}113{,}664}\left(1 - \frac{x}{4{\cdot}967{,}58}\right)^{6{\cdot}736{,}982}.$$

The origin, which is the mode, is at $16{\cdot}666{,}67 + 0{\cdot}190{,}15 = 16{\cdot}856{,}82$, whilst the range is roughly from $x = -9{\cdot}0$ to $x = 5{\cdot}0$.

The curve is shown in Fig. VII, and its ordinates are

Abscissa	Ordinate	Abscissa	Ordinate
−7·0	0·001	−0·5	256·229
−6·0	0·077	0·0	269·791
−5·0	1·420	0·5	255·379
−4·0	10·837	1·0	214·612
−3·0	45·603	2·0	96·793
−2·0	122·873	3·0	17·570
−1·0	220·297	4·0	0·256

For the representation of the complete Distribution of Means of Samples of 10 from (C) by this curve, we have

Value of Mean	Frequency in Curve	Actual Frequency
1·3 and under	17·40	15·73
1·4	55·56	54·34
1·5	135·48	136·28
1·6	227·76	231·42
1·7	261·48	269·49
1·8	206·40	199·52
1·9	80·88	80·37
2·0	15·04	12·86

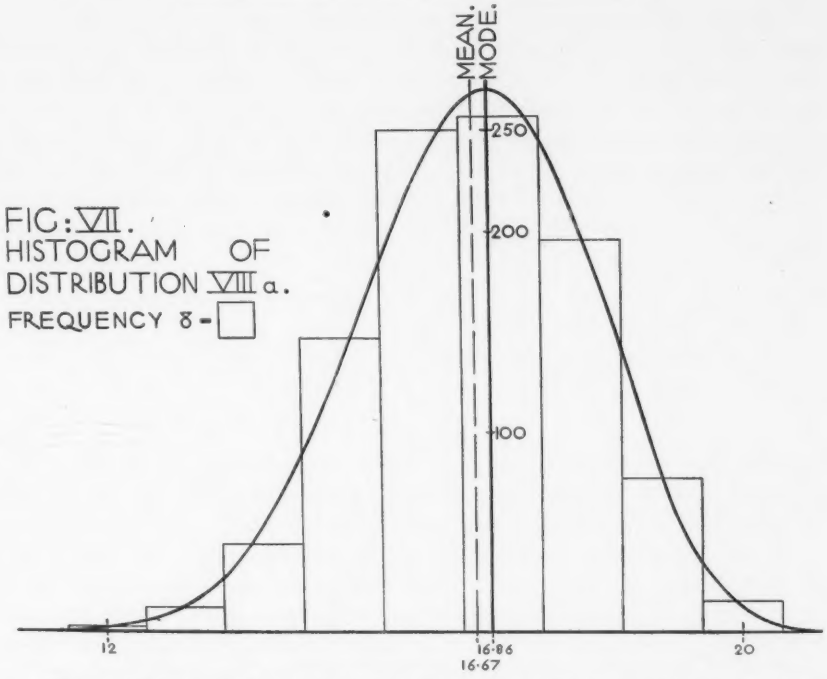


FIG: VII.
HISTOGRAM OF
DISTRIBUTION VIII a.
FREQUENCY δ = □

The small frequencies at the tails have been clubbed together so that the Goodness of Fit test may be applied. The actual frequencies are the true frequencies of the complete Distribution of Means of Samples of 10 from (C) multiplied by $\frac{1000}{254,186,856}$ to bring the total frequency down to 1000, as is necessary for comparison. This complete Distribution of Means is, as we have seen, a set of discrete frequencies, but as these discrete frequencies occur at equal intervals, we can represent it by a histogram by drawing rectangles, whose

areas are the separate discrete frequencies and whose mid-ordinates correspond to the values of the mean for which the discrete frequencies occur. Hence we can obtain a comparison between the curve and the distribution which is quite reasonable.

The application of the Goodness of Fit test gave $P = \cdot 993$ with $\chi^2 = 1 \cdot 04$ for 8 Groups. Thus, even in this extreme case, a Pearson Curve represents perfectly adequately the true Distribution of Means of Samples.

Still further, an inspection of the betas of this distribution shows that, even in this exceptional type of sampled population, the Distribution of Means approximates very closely to normality. This is so clear, especially in the light of the fits to the Normal Curve of previous distributions of means in this paper, that the determination of the actual fit is not worth the labour.

Thus it may be concluded that the Distribution of Means of Samples from practically any "finite" population can be treated for practical purposes as a normal distribution, or, if a better representation is desired, can be represented by a Pearson Curve obtained from its first four moments.

(*b*) From the previous consideration, formulae (1), (2), (3) and (4) of this part, together with a Pearson Curve, will represent adequately an actual distribution of means from a "finite" population, i.e. a distribution of means of samples where each sample was drawn completely before replacement. Now, it has been shown that the formulae

$$ {}_1M_2 = \frac{\bar{\nu}_2}{N} \quad \text{.....(5),} $$

$$ {}_1B_1 = \frac{\bar{\beta}_1}{3} \quad \text{.....(6),} $$

$$ {}_1B_2 = \frac{\bar{\beta}_2 - 3}{N} + 3 \quad \text{.....(7),} $$

together with a Pearson Curve, will likewise represent an actual distribution of means from an "infinite" population, i.e. a distribution of means where the individuals of the sample were drawn singly and replaced.

It will be useful if a practical indication can be obtained as to when the results of these two sets of formulae are not significantly different, that is, an indication as to when the Pearson Curve obtained from either set will represent reasonably well in practice, not only distributions of means obtained by sampling conducted in accordance with the assumption made in obtaining the curve, but also distributions obtained by sampling conducted in accordance with the opposite assumption. When this is so, it follows that a Pearson Curve obtained from the constants given by the simpler formulae (5), (6) and (7) will represent adequately in practice any actual distribution of means of samples obtained from sampling conducted by either method, that is, the assumption of an infinite sampled population will be legitimate, in practice, for the representation of distributions of means arising from sets of samples obtained either by drawing the complete sample, or individuals singly, before replacement.

The following tables give approximate values of ${}_1B_1$ and ${}_1B_2$ for values of M, N, $\bar{\beta}_1$ and $\bar{\beta}_2$ covering a fairly extensive range. For convenience, the value M of the total frequency of the sampled population has been expressed as a multiple of the magnitude N of the sample. In each table the row opposite $\frac{M}{N} = \infty$ gives the values of the constants for samples drawn from an infinite population, that is, the values calculated by formulae (5), (6) and (7).

Tables giving the values of ${}_1B_1$ for various values of M, N and $\bar{\beta}_1$.

$\bar{\beta}_1 = 0{\cdot}1.$

M/N	$N=2$	$N=5$	$N=10$	$N=50$	$N=100$
∞	0·050000	0·020000	0·010000	0·002000	0·001000
100	0·049242	0·019519	0·009730	0·001941	0·000970
50	0·048470	0·019036	0·009461	0·001883	0·000941
10	0·041701	0·015123	0·007330	0·001431	0·000713
5	0·031641	0·010208	0·004785	0·000911	0·000453

$\bar{\beta}_1 = 0{\cdot}5.$

M/N	$N=2$	$N=5$	$N=10$	$N=50$	$N=100$
∞	0·250000	0·100000	0·050000	0·010000	0·005000
100	0·246212	0·097595	0·048651	0·009707	0·004852
50	0·242348	0·095181	0·047304	0·009415	0·004705
10	0·208505	0·075617	0·036651	0·007154	0·003566
5	0·158203	0·051040	0·023926	0·004555	0·002264

$\bar{\beta}_1 = 1{\cdot}0.$

M/N	$N=2$	$N=5$	$N=10$	$N=50$	$N=100$
∞	0·500000	0·200000	0·100000	0·020000	0·010000
100	0·492425	0·195191	0·097302	0·019414	0·009704
50	0·484696	0·190363	0·094608	0·018831	0·009410
10	0·417010	0·151235	0·073303	0·014308	0·007133
5	0·316406	0·102079	0·047852	0·009109	0·004527

$\bar{\beta}_1 = 1{\cdot}5.$

M/N	$N=2$	$N=5$	$N=10$	$N=50$	$N=100$
∞	0·750000	0·300000	0·150000	0·030000	0·015000
100	0·738637	0·292786	0·145953	0·029121	0·014556
50	0·727044	0·285544	0·141912	0·028246	0·014115
10	0·625514	0·226852	0·109954	0·021462	0·010699
5	0·474609	0·153119	0·071777	0·013664	0·006791

$\bar{\beta}_1 = 2{\cdot}0.$

M/N	$N=2$	$N=5$	$N=10$	$N=50$	$N=100$
∞	1·000000	0·400000	0·200000	0·040000	0·020000
100	0·984849	0·390381	0·194604	0·038827	0·019408
50	0·969392	0·380726	0·189216	0·037662	0·018819
10	0·834019	0·302469	0·146606	0·028616	0·014265
5	0·632812	0·204159	0·095703	0·018218	0·009054

Tables giving the values of $_1B_2$ *for various values of* M, N *and* $\bar{\beta}_2$.

$\beta_2 = 1.$

M/N	$N=2$	$N=5$	$N=10$	$N=50$	$N=100$
∞	2·000000	2·600000	2·800000	2·960000	2·980000
100	2·010101	2·606406	2·803593	2·960780	2·980394
50	2·020408	2·612823	2·807170	2·961554	2·980784
10	2·111111	2·664303	2·835052	2·967449	2·983751
5	2·250001	2·727273	2·867021	2·973887	2·986972

$\bar{\beta}_2 = 2.$

M/N	$N=2$	$N=5$	$N=10$	$N=50$	$N=100$
∞	2·500000	2·800000	2·900000	2·980000	2·990000
100	2·497398	2·798365	2·899085	2·979802	2·989900
50	2·494586	2·796657	2·898142	2·979599	2·989798
10	2·462963	2·780142	2·889438	2·977799	2·988894
5	2·390626	2·750988	2·875166	2·975004	2·987501

$\bar{\beta}_2 = 3.$

M/N	$N=2$	$N=5$	$N=10$	$N=50$	$N=100$
∞	3·000000	3·000000	3·000000	3·000000	3·000000
100	2·984695	2·990323	2·994578	2·998823	2·999406
50	2·968763	2·980490	2·989113	2·997644	2·998811
10	2·814815	2·895981	2·943825	2·988148	2·994037
5	2·531264	2·774704	2·883311	2·976122	2·988030

$\bar{\beta}_2 = 4.$

M/N	$N=2$	$N=5$	$N=10$	$N=50$	$N=100$
∞	3·500000	3·200000	3·100000	3·020000	3·010000
100	3·471993	3·182281	3·090071	3·017844	3·008912
50	3·442940	3·164323	3·080085	3·015689	3·007824
10	3·166667	3·011820	2·998212	2·998497	2·999180
5	2·671876	2·798419	2·891456	2·977240	2·988560

$\bar{\beta}_2 = 6.$

M/N	$N=2$	$N=5$	$N=10$	$N=50$	$N=100$
∞	4·500000	3·600000	3·300000	3·060000	3·030000
100	4·444659	3·566198	3·281057	3·055887	3·027923
50	4·391295	3·531990	3·260293	3·051780	3·025851
10	3·870370	3·243499	3·106985	3·019196	3·009465
5	2·953127	2·845850	2·907746	2·979476	2·989618

To determine approximately whether a value in any column of one of these tables is significantly different or not from the value at the head of the column, we are practically compelled to use the ordinary criterion of the Probable Error. The difference between these values must be compared with the P.E. of the value considered and the difference counted significant or not according as it is of the same order as, or considerably less than, three times this P.E.

The value of the P.E. will depend, of course, upon the total frequency desired for the distribution of means to be represented, but, apart from this, the value of the P.E. is still very doubtful as it has been obtained on the assumption that the higher betas of this distribution of means are given by the well-known difference formulae referred to in previous sections of this paper. This is, in general, only very approximately true, consequently the value of the P.E. can only be taken as a rough guide. However, this very roughness makes it possible to interpolate approximately in and between the tables, concise as they are, and so to obtain rough values of ${}_1B_1$ and ${}_1B_2$ for values of M, N, $\bar{\beta}_1$ and $\bar{\beta}_2$ not explicitly occurring. These, used in conjunction with this somewhat doubtful value of the P.E., will give at any rate an indication as to whether the difference in the two values of ${}_1B_1$ or ${}_1B_2$, as the case may be, is significant or not.

It is thus probable that, by interpolation in and between these tables together with the evaluation of the P.E.* of the value of the constant so approximately obtained, a rough but fairly reliable indication can be obtained quite quickly as to whether in any given case the population sampled may be treated as infinite or not, that is, as to whether the method (complete samples or individuals drawn before replacement) of sampling is immaterial or not.

In order to obtain some confirmation of this, two particular cases were investigated:

(1) Distributions of Means of Samples from Population (A).

(2) The Distribution of Means of Samples from Population (C).

(1) *Distributions of Means of* 1000 *Samples of* 10 *from Population* (A).

Now we have the following four series of Distributions of Means for 1000 Samples of 10 from this population:

I(*a*) The Means of 1000 Samples of 10, where the actual sampling was conducted by drawing marked "tickets" singly and replacing each time.

* See *Tables for Statisticians and Biometricians*, pp. 68—71.

II (a) The Means of 1000 Samples of 10, where the actual sampling was conducted by drawing marked "tickets," the complete sample being drawn each time before replacement.

III (a) The Means of 1000 Samples of 10, where the sampling was conducted by Tippett's Random Sample.

V (a) The Means of 1000 Samples of 10, where the sampling was conducted by drawing small commercial coloured beads, the complete sample being drawn each time before replacement.

Now for Population (A) $\bar{\beta}_1 = 0{\cdot}189{,}424$, $\bar{\beta}_2 = 3{\cdot}160{,}978$, and the total frequency is 4011. Thus it is easily seen by a rough application of the tables and the use of the P.E. obtained on the basis of 1000 samples, that the differences between the values of ${}_1B_1$ and of ${}_1B_2$ are in each case non-significant.

Confirming this by actual evaluation from the formulae, we have

(a) By formulae (2), (3) and (4) of this Part (Finite Sampled Population),

$${}_1B_1 = 0{\cdot}018{,}815 \pm {\cdot}007{,}018, \quad {}_1B_2 = 3{\cdot}014{,}569 \pm {\cdot}111{,}921.$$

(b) By formulae (5), (6) and (7) of this Part (Infinite Sampled Population),

$${}_1B_1 = 0{\cdot}018{,}942 \pm {\cdot}007{,}146, \quad {}_1B_2 = 3{\cdot}016{,}098 \pm {\cdot}111{,}224.$$

Clearly the differences are in each case non-significant. We should thus expect that the Pearson Curve with betas 0·018,942 and 3·016,098 would represent not only Distributions I (a) and III (a), but also Distributions II (a) and V (a) quite adequately. This curve has been given in Part II, Section (c), its equation being

$$y = 328{\cdot}5432\left(1 + \frac{x}{54{\cdot}216{,}202}\right)^{389{\cdot}606{,}521}\left(1 - \frac{x}{13{\cdot}207{,}422}\right)^{94{\cdot}911{,}385}.$$

Now applying the Goodness of Fit test to the representation of these distributions by this curve, we have

I (a)	15 Groups,	$\chi^2 = 84{\cdot}07$,	$P = {\cdot}0000$.
II (a)	19 Groups,	$\chi^2 = 24{\cdot}28$,	$P = {\cdot}1466$.
III (a)	16 Groups,	$\chi^2 = 7{\cdot}01$,	$P = {\cdot}9573$.
V (a)	16 Groups,	$\chi^2 = 13{\cdot}74$,	$P = {\cdot}5455$.

The fit to II (a) is not very good, but it is much better than that to I (a), where the sampling was conducted in accordance with theory, and thus the poorness of fit can be ascribed principally to the difficulty of shuffling properly a large mass of tickets. The fit to V (a) is quite good, consequently two conclusions are suggested:

Firstly that, when the tables of this section indicate the differences between the values of ${}_1B_1$ and ${}_1B_2$ are clearly non-significant, the population can be treated theoretically as infinite and the method of actual sampling (complete samples or individuals drawn between replacements) is immaterial.

Secondly, when Tippett's Random Sample is inapplicable or not obtainable, the inaccuracies involved in the use of "tickets" for actual sampling arising from the great difficulty of reliable shuffling can be very considerably reduced, for large populations, by using some more symmetrical and homogeneous material like small coloured commercial beads.

(2) *The Distribution of Means of Samples of* 10 *from Population* (C).

Now the betas of (C) are $\beta_1 = 1{\cdot}6$ and $\beta_2 = 3{\cdot}6$, whilst its total frequency is 36; thus it is easily seen, by rough application of the tables and calculation of the corresponding P.E.'s on the basis of say 1000 samples, that the differences between the two values of ${}_1B_1$ and the two values of ${}_1B_2$ are distinctly significant. Therefore the complete Distribution of Means of Samples of 10 from this population obtained at the commencement of Part IV cannot be represented by a curve or distribution obtained on the assumption that the population is to be considered infinite.

For this population, as it is a binomial, not only have we the complete discrete Distribution of Means mentioned above, but we have also the discrete Distribution of Means obtained by considering the population infinite. That is, in this case, although we have seen the representation by a Pearson Curve would be adequate, it is not necessary.

Assuming (C) to be infinite, i.e. in any sampling individuals only are drawn before replacement, the Distribution of Means of 1000 Samples of 10 is the binomial*

$$1000\ (\tfrac{1}{6} + \tfrac{5}{6})^{10}, \text{ the unit being } \tfrac{1}{10} \text{ of that of (C).}$$

Expanding this, and changing the unit back to that of (C), this distribution is

Value of Mean	0·6	0·7	0·8	0·9	1·0	1·1	1·2	1·3
Frequency	·0002	·0017	·0135	·0897	·4935	2·2431	8·4116	25·8820
Value of Mean	1·4	1·5	1·6	1·7	1·8	1·9	2·0	
Frequency	64·7050	129·4100	202·2030	237·8860	198·2380	104·3360	26·0840	

The two Distributions of Means are thus

Value of the Mean	Frequency on the assumption (C) is infinite	Frequency on the assumption (C) is finite
0·6	0·0002	0·0000
0·7	0·0017	0·0000
0·8	0·0135	0·0000
0·9	0·0897	0·0000
1·0	0·4935	0·0044
1·1	2·2431	0·1426
1·2	8·4116	1·9533
1·3	25·8820	13·6272
1·4	64·7050	54·3461
1·5	129·4100	136·2833
1·6	202·2030	231·4189
1·7	237·8860	269·4910
1·8	198·2380	199·5179
1·9	104·3360	80·3745
2·0	26·0840	12·8599

* It may not be generally familiar, and it is indeed doubtful as to whether it has been published or not, although known in the Biometric Laboratory for some considerable time, that the Distribution of Means of Samples of N from an infinite population represented by the binomial $(p+q)^n$ is the binomial $p+q)^{nN}$, plotted to $1/N$ the unit of the original binomial.

In the second case the frequencies have been multiplied by $\frac{1000}{254,186,856}$ for comparison.

The fact that the first distribution in no way represents the second is obvious, but applying the Goodness of Fit test we have, for 8 Groups,

$$\chi^2 = 35{\cdot}07, \quad P = {\cdot}0000.$$

We thus see that the tables of this section, together with the use of the Probable Errors of the constants, are quite likely to give a rough but useful indication as to when a given population can be treated as infinite and the sampling conducted by either method, that is, as far as the consideration of the Distribution of Means of Samples is concerned.

(*c*) The representation of Distributions of Means of Samples, obtained as the results of actual sampling, by the Pearson Curve, whose constants are given by formulae (1), (2), (3) and (4) of this part.

Now it was seen in (*b*) that a closer agreement between theory and practice was obtained when the actual sampling was carried out by small commercial coloured beads than when "tickets" were used. It is perfectly clear, *a priori*, that for perfect shuffling perfectly homogeneous and symmetrical material is required, but, when the sampled population is large, questions of cost and total weight of material are important. Tickets, which are cheap and light, have been frequently used but the results obtained are in general quite unsatisfactory, as has been shown in previous parts of this paper. Coloured beads, when small and having merely the rough homogeneity of size and shape that obtains in cheap commercial products, have been shown to be quite satisfactory when sampling from large populations, and moreover the variety of shades of colour enables them to be used when the sampled population, like Population (A), has a large number of categories.

Coming now to actual sampling from small populations, it seemed clear that "tickets" would be unsuitable, so an attempt was made to use the same beads as had been used with success when sampling from the large Population (A).

Population (C) has three categories *A*, *B* and *C*, of frequencies 1, 10 and 25, and these were represented by the appropriate number of red, orange and yellow beads taken from the 4011 used for Population (A). These 36 beads, it may be remarked, were by no means uniform in shape; they were none of them really spherical, and since their weights varied from ·512 g. to ·864 g., so there was also little uniformity of size and weight.

Before giving the results of the actual sets of samples made, it may be well to re-state the values of the constants of the theoretic Distribution of Means as given by the formulae. They are

$$\text{Mean} = 1{\cdot}666,667, \qquad \Sigma_1 = 0{\cdot}143,649,$$
$$_1B_1 = 0{\cdot}047,697, \qquad _1B_2 = 2{\cdot}816,989,$$

and the Pearson Curve with these constants is

$$y_0 = 269{\cdot}791\left(1+\frac{x}{8{\cdot}932{,}13}\right)^{12{\cdot}113{,}664}\left(1-\frac{x}{4{\cdot}967{,}58}\right)^{6{\cdot}736{,}982},$$

whose ordinates etc. have been given before.

Distribution VI (*a*). The Means of 1000 Samples of 10 from Population (C) obtained by drawing beads from a bowl, the complete sample being drawn before replacement.

Value of Mean	*1·2*	*1·3*	*1·4*	*1·5*	*1·6*	*1·7*	*1·8*	*1·9*	*2·0*
Frequency	5	13	74	197	241	226	163	72	9

The constants are as follows, the origin being 1·6 and the working unit 10 times that of the distribution, i.e. 10 times that of Population (C):

$$\begin{aligned}
\nu_1' &= 0{\cdot}400{,}000, & \nu_2 &= 2{\cdot}200{,}000, & \text{Mean} &= 1{\cdot}640{,}000,\\
\nu_2' &= 2{\cdot}360{,}000, & \nu_3 &= -0{\cdot}114{,}000, & \Sigma_1 &= 0{\cdot}148{,}324,\\
\nu_3' &= 2{\cdot}590{,}000, & \nu_4 &= 12{\cdot}728{,}800, & {}_1B_1 &= 0{\cdot}001{,}221,\\
\nu_4' &= 14{\cdot}684{,}000, & & & {}_1B_2 &= 2{\cdot}629{,}917.
\end{aligned}$$

Distribution VII (*a*). The Means of 1000 Samples of 10 from Population (C) drawn as in VI (*a*), but a *different* set of 36 beads being used. These beads were, however, merely another 36 from the 4011 used in V (*a*).

Value of Mean	*1·2*	*1·3*	*1·4*	*1·5*	*1·6*	*1·7*	*1·8*	*1·9*	*2·0*
Frequency	4	7	43	108	244	276	204	99	15

The constants are (working unit as before):

$$\begin{aligned}
\nu_1' &= 0{\cdot}810{,}000, & \nu_2 &= 1{\cdot}973{,}900, & \text{Mean} &= 1{\cdot}681{,}000,\\
\nu_2' &= 2{\cdot}630{,}000, & \nu_3 &= -0{\cdot}684{,}018, & \Sigma_1 &= 0{\cdot}140{,}496,\\
\nu_3' &= 4{\cdot}644{,}000, & \nu_4 &= 11{\cdot}801{,}297, & {}_1B_1 &= 0{\cdot}060{,}836,\\
\nu_4' &= 17{\cdot}786{,}000, & & & {}_1B_2 &= 3{\cdot}028{,}862.
\end{aligned}$$

These are likewise discrete distributions, but as the intervals are equal, they were represented, as in the previous case, by histograms and tested for Goodness of Fit with the curve $y = 269{\cdot}791\left(1+\frac{x}{8{\cdot}932{,}13}\right)^{12{\cdot}113{,}664}\left(1-\frac{x}{4{\cdot}967{,}58}\right)^{6{\cdot}736{,}982}$ given above.

The results were

VI (*a*) $\chi^2 = 52{\cdot}18$, $P = {\cdot}000{,}000$ with 8 Groups.

VII (*a*) $\chi^2 = 16{\cdot}82$, $P = {\cdot}018{,}786$ with 8 Groups.

The fits here are very bad.

A glance at the values of the constants of VI (*a*) and VII (*a*) will show that these poor fits are largely due to displacements of the means of these distributions from the theoretical value 1·666,667. Now each of these two distributions was obtained by taking four sets of 250 samples, each 250 samples being completed at one sitting. When the fits were discovered to be poor owing to displacements of the means, the separate means of these 8 sets of 250 samples were evaluated with the following results:

VI (*a*)	1st set of 250 Samples	Mean = 1·6448.
	2nd „ „ „ „	Mean = 1·6440.
	3rd „ „ „ „	Mean = 1·6260.
	4th „ „ „ „	Mean = 1·6452.
VII (*a*) (Different set	1st „ „ „ „	Mean = 1·6740.
of beads)	2nd „ „ „ „	Mean = 1·6916.
	3rd „ „ „ „	Mean = 1·6724.
	4th „ „ „ „	Mean = 1·6860.

Now the P.E. of the Mean 1·666,667 on the basis of 1000 samples is 0·003,064, consequently it is clear that with the set of beads used in VI (*a*) the mean was consistently significantly below its true value, whilst with the set used in VII (*a*) the mean was consistently significantly above its true value. It is clear therefore that the lack of homogeneity of shape and size of the beads tends to produce a definite bias in the sampling, and this bias will depend on the particular beads chosen.

We may thus conclude that, although the beads may be sufficiently homogeneous to produce effective sampling from a large population, yet when samples are taken from a small population, material which is very close to the ideal of complete homogeneity of size and weight together with complete symmetry of shape is absolutely requisite.

To this end 1000 fresh samples were made with clay marbles as the medium instead of beads; steel spheres would have been ideal as regards the conditions of uniformity of size and shape, but they would have been difficult to mark so that the categories of the population were readily distinguishable. These marbles were not quite of uniform size and weight (their weights ranged from 3·1535 g. to 3·8955 g.), but they were much more spherical and therefore much more uniform in shape than the beads.

Distribution VIII (*a*). The Means of 1000 Samples of 10 from Population (C) drawn as in VI (*a*), but by the use of coloured spherical clay marbles.

Value of Mean	*1·2*	*1·3*	*1·4*	*1·5*	*1·6*	*1·7*	*1·8*	*1·9*	*2·0*
Frequency	2	12	44	147	250	257	196	77	15

The constants are (working unit as before):

$\nu_1' = 0{\cdot}661{,}000,$	$\nu_2 = 2{\cdot}000{,}079,$	Mean $= 1{\cdot}666{,}100,$
$\nu_2' = 2{\cdot}437{,}000,$	$\nu_3 = -0{\cdot}341{,}961,$	$\Sigma_1 = 0{\cdot}141{,}424,$
$\nu_3' = 3{\cdot}913{,}000,$	$\nu_4 = 11{\cdot}274{,}984,$	$_1B_1 = 0{\cdot}014{,}615,$
$\nu_4' = 15{\cdot}805{,}000,$		$_1B_2 = 2{\cdot}818{,}523.$

Representing this discrete distribution by a histogram (the intervals are equal) and testing for the Goodness of Fit of the representation by the curve

$$y = 269{\cdot}791\left(1 + \frac{x}{8{\cdot}932{,}13}\right)^{12{\cdot}113{,}084}\left(1 - \frac{x}{4{\cdot}967{,}58}\right)^{6{\cdot}736{,}982},$$

the result was

VIII (*a*) $\chi^2 = 7{\cdot}01$, $P = {\cdot}4288$ with 8 Groups.

The fit here (shown in Fig. VII, see p. 359) is quite reasonable, but doubtless could be improved by performing the sampling with even more homogeneous and symmetrical material.

Distribution VIII (*a*), being an experimental distribution of Means of Samples obtained by actual sampling from (C), can be fitted to the true Distribution of Means given at the commencement of this part, as a verification of the efficacy of this material.

The fit was $\chi^2 = 5{\cdot}63$, $P = {\cdot}5842$ with 8 Groups, providing a satisfactory comparison, but in general practice the true Distribution of Means would be unknown and the comparison could not be carried out, at least not until our knowledge is extended to the direct establishment of the Distribution of Means from any finite population.

We see thus that for sampling from a small population, if there is to be any good agreement between theory and its representation as regards the distribution of means obtained from actual sets of samples, the material employed must approximate closely to the ideal of complete homogeneity of shape, size and weight. This material must be chosen therefore with more care than perhaps has been deemed necessary in the past; for instance, I am convinced, cardboard tickets are quite useless.

PART V. THE DISTRIBUTION OF σ^2 OF SAMPLES OF N FROM A POPULATION OF M.

The Mean and Second Moment of this Distribution have been given by Dr Neyman in the paper to which previous reference has been made, but he did not attempt the much more laborious task of the determination of the Third and Fourth Moment-Coefficients, which are necessary if the distribution is to be represented by a Pearson Curve.

Dr Neyman's results are

$$ {}_2M_1' = \frac{(N-1)\,M}{N\,(M-1)}\,\bar{\nu}_2 \quad \text{.....(1),} $$

$$ {}_2M_2 = \frac{M\,(M-N)\,(N-1)\,\bar{\nu}_2^{\,2}}{N^3\,(M-1)^2\,(M-2)\,(M-3)}\,[(MN-M-N-1)\,(M-1)\,\bar{\beta}_2 - (M^2N-3M^2+6M-3N-3)] \quad \text{.....(2).} $$

To obtain the Third and Fourth Moments ${}_2M_3$ and ${}_2M_4$ the following expansions are required in addition to those given in *Biometrika*, Vol. XVII. p. 80.

S being the summation over any finite distribution of X's we have, for the evaluation of ${}_2M_3$:

$$
\begin{aligned}
S(X_1^5)\,S(X_1) &= S(X_1^6) + S(X_1^5X_2),\\
S(X_1^4)\,S(X_1^2) &= S(X_1^6) + S(X_1^4X_2^2),\\
S(X_1^4)\,\{S(X_1)\}^2 &= S(X_1^6) + 2S(X_1^5X_2) + S(X_1^4X_2^2) + 2S(X_1^4X_2X_3),\\
S(X_1^3)\,\{S(X_1)\}^3 &= S(X_1^6) + 3S(X_1^5X_2) + 3S(X_1^4X_2^2) + 6S(X_1^4X_2X_3)\\
&\quad + 2S(X_1^3X_2^3) + 3S(X_1^3X_2^2X_3) + 6S(X_1^3X_2X_3X_4),\\
\{S(X_1^3)\}^2 &= S(X_1^6) + 2S(X_1^3X_2^3),\\
\{S(X_1^2)\}^2\,\{S(X_1)\}^2 &= S(X_1^6) + 2S(X_1^5X_2) + 3S(X_1^4X_2^2) + 2S(X_1^4X_2X_3)\\
&\quad + 4S(X_1^3X_2^3) + 4S(X_1^3X_2^2X_3) + 6S(X_1^2X_2^2X_3^2)\\
&\quad + 4S(X_1^2X_2^3X_3X_4),\\
S(X_1^2)\,\{S(X_1)\}^4 &= S(X_1^6) + 4S(X_1^5X_2) + 7S(X_1^4X_2^2) + 12S(X_1^4X_2X_3)\\
&\quad + 8S(X_1^3X_2^3) + 16S(X_1^3X_2^2X_3) + 24S(X_1^3X_2X_3X_4)\\
&\quad + 18S(X_1^2X_2^2X_3^2) + 24S(X_1^2X_2^2X_3X_4)\\
&\quad + 24S(X_1^2X_2X_3X_4X_5),\\
S(X_1)\,S(X_1^2)\,S(X_1^3) &= S(X_1^6) + S(X_1^5X_2) + S(X_1^4X_2^2) + 2S(X_1^3X_2^3)\\
&\quad + S(X_1^3X_2^2X_3),\\
S(X_1^7)\,S(X_1) &= S(X_1^8) + S(X_1^7X_2),\\
S(X_1^6)\,S(X_1^2) &= S(X_1^8) + S(X_1^6X_2^2),\\
S(X_1^6)\,\{S(X_1)\}^2 &= S(X_1^8) + 2S(X_1^7X_2) + S(X_1^6X_2^2) + 2S(X_1^6X_2X_3),\\
S(X_1^5)\,S(X_1^3) &= S(X_1^8) + S(X_1^5X_2^3),\\
S(X_1^5)\,S(X_1^2)\,S(X_1) &= S(X_1^8) + S(X_1^7X_2) + S(X_1^6X_2^2) + S(X_1^5X_2^3)\\
&\quad + S(X_1^5X_2^2X_3),\\
S(X_1^5)\,\{S(X_1)\}^3 &= S(X_1^8) + 3S(X_1^7X_2) + 3S(X_1^6X_2^2) + 6S(X_1^6X_2X_3)\\
&\quad + S(X_1^5X_2^3) + 3S(X_1^5X_2^2X_3) + 6S(X_1^5X_2X_3X_4),\\
\{S(X_1^4)\}^2 &= S(X_1^8) + 2S(X_1^4X_2^4),\\
S(X_1^4)\,S(X_1^3)\,S(X_1) &= S(X_1^8) + S(X_1^7X_2) + S(X_1^5X_2^3) + 2S(X_1^4X_2^4)\\
&\quad + S(X_1^4X_2^3X_3),\\
S(X_1^4)\,\{S(X_1^2)\}^2 &= S(X_1^8) + 2S(X_1^6X_2^2) + 2S(X_1^4X_2^4) + 2S(X_1^4X_2^2X_3^2),
\end{aligned}
$$

$$
\begin{aligned}
S(X_1^4)\,S(X_1^2)\,\{S(X_1)\}^2 = {} & S(X_1^8) + 2S(X_1^7X_2) + 2S(X_1^6X_2^2) + 2S(X_1^6X_2X_3) \\
& + 2S(X_1^5X_2^3) + 2S(X_1^5X_2^2X_3) + 2S(X_1^4X_2^4) \\
& + 2S(X_1^4X_2^3X_3) + 2S(X_1^4X_2^2X_3^2) + 2S(X_1^4X_2^2X_3X_4), \\
S(X_1^4)\,\{S(X_1)\}^4 = {} & S(X_1^8) + 4S(X_1^7X_2) + 6S(X_1^6X_2^2) + 12S(X_1^6X_2X_3) \\
& + 4S(X_1^5X_2^3) + 12S(X_1^5X_2^2X_3) + 24S(X_1^5X_2X_3X_4) \\
& + 2S(X_1^4X_2^4) + 4S(X_1^4X_2^3X_3) + 6S(X_1^4X_2^2X_3^2) \\
& + 12S(X_1^4X_2^2X_3X_4) + 24S(X_1^4X_2X_3X_4X_5), \\
\{S(X_1^3)\}^2\,S(X_1^2) = {} & S(X_1^8) + S(X_1^6X_2^2) + 2S(X_1^5X_2^3) + 2S(X_1^3X_2^3X_3^2), \\
\{S(X_1^3)\}^2\,\{S(X_1)\}^2 = {} & S(X_1^8) + 2S(X_1^7X_2) + S(X_1^6X_2^2) + 2S(X_1^6X_2X_3) \\
& + 2S(X_1^5X_2^3) + 4S(X_1^4X_2^4) + 4S(X_1^4X_2^3X_3) \\
& + 2S(X_1^3X_2^3X_3^2) + 4S(X_1^3X_2^3X_3X_4), \\
S(X_1^3)\,\{S(X_1^2)\}^2\,S(X_1) = {} & S(X_1^8) + S(X_1^7X_2) + 2S(X_1^6X_2^2) + 3S(X_1^5X_2^3) \\
& + 2S(X_1^5X_2^2X_3) + 2S(X_1^4X_2^4) + S(X_1^4X_2^3X_3) \\
& + 2S(X_1^4X_2^2X_3^2) + 4S(X_1^3X_2^3X_3^2) + 2S(X_1^3X_2^2X_3^2X_4), \\
S(X_1^3)\,S(X_1^2)\,\{S(X_1)\}^3 = {} & S(X_1^8) + 3S(X_1^7X_2) + 4S(X_1^6X_2^2) + 6S(X_1^6X_2X_3) \\
& + 5S(X_1^5X_2^3) + 6S(X_1^5X_2^2X_3) + 6S(X_1^5X_2X_3X_4) \\
& + 6S(X_1^4X_2^4) + 9S(X_1^4X_2^3X_3) + 6S(X_1^4X_2^2X_3^2) \\
& + 6S(X_1^4X_2^2X_3X_4) + 8S(X_1^3X_2^3X_3^2) \\
& + 12S(X_1^3X_2^3X_3X_4) + 6S(X_1^3X_2^2X_3^2X_4) \\
& + 6S(X_1^3X_2^2X_3X_4X_5), \\
S(X_1^3)\,\{S(X_1)\}^5 = {} & S(X_1^8) + 5S(X_1^7X_2) + 10S(X_1^6X_2^2) + 20S(X_1^6X_2X_3) \\
& + 11S(X_1^5X_2^3) + 30S(X_1^5X_2^2X_3) + 60S(X_1^5X_2X_3X_4) \\
& + 10S(X_1^4X_2^4) + 25S(X_1^4X_2^3X_3) + 30S(X_1^4X_2^2X_3^2) \\
& + 60S(X_1^4X_2^2X_3X_4) + 120S(X_1^4X_2X_3X_4X_5) \\
& + 20S(X_1^3X_2^3X_3^2) + 40S(X_1^3X_2^3X_3X_4) + 30S(X_1^3X_2^2X_3^2X_4) \\
& + 60S(X_1^3X_2^2X_3X_4X_5) + 120S(X_1^3X_2X_3X_4X_5X_6).
\end{aligned}
$$

The Third and Fourth Moment-Coefficients of the Distribution of σ^2 of Samples of N from a population of M.

For this we have

$$\frac{M!}{N!\,(M-N)!}\,{}_2M_3' = \Sigma\left[\frac{S(X_1^3)}{N} - \left\{\frac{S(X_1)}{N}\right\}^3\right]^3,$$

where S represents the summation over the N values of the variate forming the sample and Σ represents the summation for the $\dfrac{M!}{N!\,(M-N)!}$ samples, the sampled population $X_1X_2 \ldots X_M$ being referred to its Mean as origin.

Therefore
$$\frac{M!}{N!\,(M-N)!}\,{}_2M_3' = \Sigma\left[\frac{\{S(X_1^3)\}^3}{N^3} - 3\,\frac{\{S(X_1^2)\}^2\,\{S(X_1)\}^2}{N^4} + \frac{3S(X_1^2)\,\{S(X_1)\}^4}{N^5} - \frac{\{S(X_1)\}^6}{N^6}\right].$$

Using the appropriate expansions from the above and from *Biometrika*, Vol. XVII. p. 80, we have

$$N^6\frac{M!}{N!(M-N)!}\,{}_2M_3' = (N-1)^3\Sigma S(X_1^6) - 6(N-1)^2\Sigma S(X_1^5X_2)$$
$$+3(N-1)[(N-1)^2+4]\Sigma S(X_1^4X_2^2)$$
$$-6(N-1)(N-5)\Sigma S(X_1^4X_2X_3) - 4[3(N-1)^2+2]\Sigma S(X_1^3X_2^3)$$
$$-12[(N-2)^2+1]\Sigma S(X_1^3X_2^2X_3) + 24[3(N-2)+1]\Sigma S(X_1^3X_2X_3X_4)$$
$$+6[(N-1)^3+6(N-3)+4]\Sigma S(X_1^2X_2^2X_3^2) - 12[(N-3)^2+6]\Sigma S(X_1^2X_2^2X_3X_4)$$
$$+72(N-5)\Sigma S(X_1^2X_2X_3X_4X_5) - 720\Sigma S(X_1X_2X_3X_4X_5X_6).$$

Now taking any product $X_1^{\alpha}X_2^{\beta}\dots X_r^{\lambda}$, the number of samples in which it will occur is clearly ${}^{N-r}C_{M-r} = \dfrac{(M-r)!}{(N-r)!\,(M-N)!}$; therefore

$$\Sigma S(X_1^{\alpha}X_2^{\beta}\dots X_r^{\lambda}) = \frac{(M-r)!}{(N-r)!\,(M-N)!}\,S(X_1^{\alpha}X_2^{\beta}\dots X_r^{\lambda}),$$

where S is now the summation over the sampled population $X_1X_2\dots X_M$.

Therefore we have

$${}_2M_3' = \frac{1}{N^5\lfloor\underline{M}}\Big\{(N-1)^3\lfloor\underline{M-1}\,S(X_1^6) - 6(N-1)^3\lfloor\underline{M-2}\,S(X_1^5X_2)$$
$$+3(N-1)^2[(N-1)^2+4]\lfloor\underline{M-2}\,S(X_1^4X_2^2)$$
$$-6(N-1)^2(N-2)(N-5)\lfloor\underline{M-3}\,S(X_1^4X_2X_3)$$
$$-4(N-1)[3(N-1)^2+2]\lfloor\underline{M-2}\,S(X_1^3X_2^3)$$
$$-12(N-1)(N-2)[(N-2)^2+1]\lfloor\underline{M-3}\,S(X_1^3X_2^2X_3)$$
$$+24(N-1)(N-2)(N-3)[3(N-2)^2+1]\lfloor\underline{M-4}\,S(X_1^3X_2X_3X_4)$$
$$+6(N-1)(N-2)[(N-1)^3+6(N-3)+4]\lfloor\underline{M-3}\,S(X_1^2X_2^2X_3^2)$$
$$-12(N-1)(N-2)(N-3)[(N-3)^2+6]\lfloor\underline{M-4}\,S(X_1^2X_2^2X_3X_4)$$
$$+72(N-1)(N-2)(N-3)(N-4)(N-5)\lfloor\underline{M-5}\,S(X_1^2X_2X_3X_4X_5)$$
$$-720(N-1)(N-2)(N-3)(N-4)(N-5)\lfloor\underline{M-6}\,S(X_1X_2X_3X_4X_5X_6)\Big\},$$

where S is the summation over the sampled population $X_1X_2\dots X_M$.

Now since the origin is the mean of this sampled population we have $S(X_1)=0$, and thus the following equations:

$$\begin{aligned}
M\bar{\nu}_6 &= S(X_1^6),\\
0 &= S(X_1^5)\,S(X_1),\\
M^2\bar{\nu}_4\bar{\nu}_2 &= S(X_1^4)\,S(X_1^2),\\
0 &= S(X_1^4)\,\{S(X_1)\}^2,\\
M^2\bar{\nu}_2^3 &= \{S(X_1^3)\}^2,\\
M^3\bar{\nu}_2^3 &= \{S(X_1^2)\}^3,\\
0 &= S(X_1^3)\,S(X_1^2)\,S(X_1),\\
0 &= S(X_1^3)\,\{S(X_1)\}^3,\\
0 &= \{S(X_1^2)\}^2\,\{S(X_1)\}^2,\\
0 &= S(X_1^2)\,\{S(X_1)\}^4,\\
0 &= \{S(X_1)\}^6,
\end{aligned}$$

where $\bar{\nu}_s$ is the sth uncorrected moment about its own mean of the sampled population.

Expanding the right-hand sides of these equations by use of the expansions above and in *Biometrika*, Vol. XVII. p. 80, we obtain 11 linear equations to express the 11 sums

$$S(X_1^6),\ S(X_1^5X_2),\ S(X_1^4X_2^2),\ S(X_1^4X_2X_3),\ S(X_1^3X_2^3),\ S(X_1^3X_2^2X_3),$$
$$S(X_1^3X_2X_3X_4),\ S(X_1^2X_2^2X_3^2),\ S(X_1^2X_2^2X_3X_4),\ S(X_1^2X_2X_3X_4X_5),$$

and $$S(X_1X_2X_3X_4X_5X_6)$$

in terms of $\bar{\nu}_6, \bar{\nu}_4\bar{\nu}_2, \bar{\nu}_3^2$ and $\bar{\nu}_2^3$.

Solving these we have, after considerable manipulation:

$$\begin{aligned}
S(X_1^6) &= M\bar{\nu}_6,\\
S(X_1^5X_2) &= -M\bar{\nu}_6,\\
S(X_1^4X_2^2) &= M^2\bar{\nu}_4\bar{\nu}_2 - M\bar{\nu}_6,\\
2S(X_1^4X_2X_3) &= 2M\bar{\nu}_6 - M^2\bar{\nu}_4\bar{\nu}_2,\\
2S(X_1^3X_2^3) &= M^2\bar{\nu}_3^2 - M\bar{\nu}_6,\\
S(X_1^3X_2^2X_3) &= 2M\bar{\nu}_6 - M^2\bar{\nu}_4\bar{\nu}_2 - M^2\bar{\nu}_3^2,\\
6S(X_1^3X_2X_3X_4) &= 3M^2\bar{\nu}_4\bar{\nu}_2 + 2M^2\bar{\nu}_3^2 - 6M\bar{\nu}_6,\\
6S(X_1^2X_2^2X_3^2) &= 2M\bar{\nu}_6 + M^3\bar{\nu}_2^3 - 3M^2\bar{\nu}_4\bar{\nu}_2,\\
4S(X_1^3X_2^2X_3X_4) &= 5M^2\bar{\nu}_4\bar{\nu}_2 + 2M^2\bar{\nu}_3^2 - M^3\bar{\nu}_2^3 - 6M\bar{\nu}_6,\\
24S(X_1^3X_2X_3X_4X_5) &= 24M\bar{\nu}_6 - 18M^2\bar{\nu}_2\bar{\nu}_4 - 8M^2\bar{\nu}_3^2 + 3M^3\bar{\nu}_2^3,\\
720S(X_1X_2X_3X_4X_5X_6) &= 90M^2\bar{\nu}_4\bar{\nu}_2 + 40M^2\bar{\nu}_3^2 - 15M^3\bar{\nu}_2^3 - 120M\bar{\nu}_6.
\end{aligned}$$

Substituting the values for these sums in the last expression for ${}_2M_3'$ we obtain:

$$\begin{aligned}
{}_2M_3' = \frac{N-1}{N^5}\Bigg\{&\bar{\nu}_6\Bigg[(N-1)^2 - \frac{(3N^3-21N^2+45N-31)}{M-1}\\
&+\frac{2(N-2)(N-3)(N^2-15N+30)}{(M-1)(M-2)} + \frac{6(N-2)(N-3)(3N^2-30N+65)}{(M-1)(M-2)(M-3)}\\
&+\frac{72(N-2)(N-3)(N-4)(N-5)}{(M-1)(M-2)(M-3)(M-4)} + \frac{120(N-2)(N-3)(N-4)(N-5)}{(M-1)(M-2)(M-3)(M-4)(M-5)}\Bigg]\\
&+\frac{M\bar{\nu}_4\bar{\nu}_2}{M-1}\Bigg[3(N-1)(N^2-2N+5) - \frac{3(N-2)(N^3-8N^2+31N-40)}{M-2}\\
&\quad-\frac{3(N-2)(N-3)(5N^2-42N+95)}{(M-2)(M-3)} - \frac{54(N-2)(N-3)(N-4)(N-5)}{(M-2)(M-3)(M-4)}\\
&\quad-\frac{90(N-2)(N-3)(N-4)(N-5)}{(M-2)(M-3)(M-4)(M-5)}\Bigg]\\
&-\frac{M\bar{\nu}_3^2}{M-1}\Bigg[2(3N^2-6N+5) - \frac{12(N-2)(N^2-4N+5)}{M-2}\\
&\quad+\frac{2(N-2)(N-3)(3N^2-30N+65)}{(M-2)(M-3)} + \frac{24(N-2)(N-3)(N-4)(N-5)}{(M-2)(M-3)(M-4)}\\
&\quad+\frac{40(N-2)(N-3)(N-4)(N-5)}{(M-2)(M-3)(M-4)(M-5)}\Bigg]
\end{aligned}$$

$$+\frac{M^2\bar{\nu}_2^3}{(M-1)(M-2)}\left[(N-2)(N^3-3N^2+9N-15)+\frac{3(N-2)(N-3)(N^2-6N+15)}{M-3}\right.$$
$$\left.+\frac{9(N-2)(N-3)(N-4)(N-5)}{(M-3)(M-4)}+\frac{15(N-2)(N-3)(N-4)(N-5)}{(M-3)(M-4)(M-5)}\right]\Bigg\}.$$

Transferring the origin to the mean of this distribution of σ^2 of samples by ${}_2M_3={}_2M_3'-3\,.\,{}_2M_2\,.\,{}_2M_1'-({}_2M_1')^3$, we have, using Neyman's results (1) and (2):

$${}_2M_3=\frac{N-1}{N^5}\left\{\bar{\nu}_6\left[(N-1)^2-\frac{(3N^3-21N^2+45N-31)}{M-1}\right.\right.$$
$$+\frac{2(N-2)(N-3)(N^2-15N+30)}{(M-1)(M-2)}+\frac{6(N-2)(N-3)(3N^2-30N+65)}{(M-1)(M-2)(M-3)}$$
$$\left.+\frac{72(N-2)(N-3)(N-4)(N-5)}{(M-1)(M-2)(M-3)(M-4)}+\frac{120(N-2)(N-3)(N-4)(N-5)}{(M-1)(M-2)(M-3)(M-4)(M-5)}\right]$$
$$-\frac{3M\bar{\nu}_4\bar{\nu}_2}{M-1}\left[(N-1)(N-5)-\frac{N(N-1)(N^2-6N+7)}{M-1}\right.$$
$$+\frac{(N-2)(N^3-8N^2+31N-40)}{M-2}-\frac{4N(N-1)(N-2)(N-3)}{(M-1)(M-2)}$$
$$\left.+\frac{(N-2)(N-3)(5N^2-42N+95)}{(M-2)(M-3)}-\frac{6N(N-1)(N-2)(N-3)}{(M-1)(M-2)(M-3)}\right.$$
$$\left.+\frac{18(N-2)(N-3)(N-4)(N-5)}{(M-2)(M-3)(M-4)}+\frac{30(N-2)(N-3)(N-4)(N-5)}{(M-2)(M-3)(M-4)(M-5)}\right]$$
$$-\frac{2M\bar{\nu}_3^2}{M-1}\left[(3N^2-6N+5)-\frac{6(N-2)(N^2-4N+5)}{M-2}\right.$$
$$+\frac{(N-2)(N-3)(3N^2-30N+65)}{(M-2)(M-3)}+\frac{12(N-2)(N-3)(N-4)(N-5)}{(M-2)(M-3)(M-4)}$$
$$\left.+\frac{20(N-2)(N-3)(N-4)(N-5)}{(M-2)(M-3)(M-4)(M-5)}\right]$$
$$-\frac{M^2\bar{\nu}_2^3}{M-1}\left[\frac{N(N-1)(N^2-4N+9)}{M-1}-\frac{(N-2)(N^3-3N^2+9N-15)}{M-2}-\frac{2N^2(N-1)^2}{(M-1)^2}\right.$$
$$+\frac{6N(N-1)(N-2)(N-3)}{(M-1)(M-2)}-\frac{3(N-2)(N-3)(N^2-6N+15)}{(M-2)(M-3)}$$
$$+\frac{9N(N-1)(N-2)(N-3)}{(M-1)(M-2)(M-3)}-\frac{9(N-2)(N-3)(N-4)(N-5)}{(M-2)(M-3)(M-4)}$$
$$\left.-\frac{15(N-2)(N-3)(N-4)(N-5)}{(M-2)(M-3)(M-4)(M-5)}\right]\Bigg\}\qquad\text{.....(3).}$$

Again

$$\frac{M!}{N!(M-N)!}{}_2M_4'=\Sigma\left[\frac{S(X_1^2)}{N}-\frac{\{S(X_1)\}^2}{N^2}\right]^4$$
$$=\Sigma\left[\frac{\{S(X_1^2)\}^4}{N^4}-\frac{4\{S(X_1^2)\}^3\{S(X_1)\}^2}{N^5}+\frac{6\{S(X_1^2)\}^2\{S(X_1)\}^4}{N^6}\right.$$
$$\left.-\frac{4\{S(X_1^2)\}\{S(X_1)\}^6}{N^7}+\frac{\{S(X_1)\}^8}{N^8}\right].$$

Using again the appropriate expansions from those at the commencement of this portion of the work and from those in *Biometrika*, Vol. XVII. p. 80, we have, after simplification:

$$
\begin{aligned}
N^8 \frac{M!}{N!(M-N)!} \cdot {}_2M_4' = \Sigma \{ & S(X_1^8)[N^4 - 4N^3 + 6N^2 - 4N + 1] \\
& + S(X_1^7 X_2)[-8N^3 + 24N^2 - 24N + 8] \\
& + S(X_1^6 X_2^2)[4N^4 - 16N^3 + 48N^2 - 64N + 28] \\
& + S(X_1^6 X_2 X_3)[-8N^3 + 72N^2 - 120N + 56] \\
& + S(X_1^5 X_2^3)[-24N^3 + 72N^2 - 104N + 56] \\
& + S(X_1^5 X_2^2 X_3)[-24N^3 + 120N^2 - 264N + 168] \\
& + S(X_1^5 X_2 X_3 X_4)[144N^2 - 480N + 336] \\
& + S(X_1^4 X_2^4)[6N^4 - 24N^3 + 84N^2 - 120N + 70] \\
& + S(X_1^4 X_2^2 X_3^2)[12N^4 - 48N^3 + 192N^2 - 480N + 420] \\
& + S(X_1^4 X_2^3 X_3)[-24N^3 + 168N^2 - 360N + 280] \\
& + S(X_1^4 X_2^2 X_3 X_4)[-24N^3 + 216N^2 - 840N + 840] \\
& + S(X_1^4 X_2 X_3 X_4 X_5)[144N^3 - 1440N + 1680] \\
& + S(X_1^3 X_2^3 X_3^2)[-48N^3 + 240N^2 - 560N + 560] \\
& + S(X_1^3 X_2^2 X_3^2 X_4)[-48N^3 + 336N^2 - 1200N + 1680] \\
& + S(X_1^3 X_2^2 X_3 X_4)[288N^2 - 960N + 1120] \\
& + (X_1^3 X_2^2 X_3 X_4 X_5)[288N^2 - 1920N + 3360] \\
& + S(X_1^3 X_2 X_3 X_4 X_5 X_6)[-2880N + 6720] \\
& + S(X_1^2 X_2^2 X_3^2 X_4^2)[24N^4 - 96N^3 + 432N^2 - 1440N + 2520] \\
& + S(X_1^2 X_2^2 X_3^2 X_4 X_5)[-48N^3 + 432N^2 - 2160N + 5040] \\
& + S(X_1^2 X_2^2 X_3 X_4 X_5 X_6)[288N^2 - 2880N + 10080] \\
& + S(X_1^2 X_2 X_3 X_4 X_5 X_6 X_7)[-2880N + 20160] \\
& + S(X_1 X_2 X_3 X_4 X_5 X_6 X_7 X_8) \cdot 40320\}.
\end{aligned}
$$

Again, since $\Sigma S(X_1^\alpha X_2^\beta \dots X_r^\lambda) = \dfrac{(M-r)!}{(N-r)!\,(M-N)!} S(X_1^\alpha X_2^\beta \dots X_r^\lambda)$, where S is now the summation over the sampled population $X_1 X_2 \dots X_M$, we have

$$
\begin{aligned}
{}_2M_4' = \frac{N-1}{N^7 \lfloor M} & \{S(X_1^8)(N-1)^3 \lfloor M-1 \\
& + S(X_1^7 X_2)(-8N^3 + 24N^2 - 24N + 8) \lfloor M-2 \\
& + S(X_1^6 X_2^2)(4N^4 - 16N^3 + 48N^2 - 64N + 28) \lfloor M-2 \\
& + S(X_1^6 X_2 X_3)(N-2)(-8N^3 + 72N^2 - 120N + 56) \lfloor M-3 \\
& + S(X_1^5 X_2^3)(-24N^3 + 72N^2 - 104N + 56) \lfloor M-2 \\
& + S(X_1^5 X_2^2 X_3)(N-2)(-24N^3 + 120N^2 - 264N + 168) \lfloor M-3
\end{aligned}
$$

$$+S(X_1^5X_2X_3X_4)(N-2)(N-3)(144N^2-480N+336)\,\underline{|M-4}$$
$$+S(X_1^4X_2^4)(6N^4-24N^3+84N^2-120N+70)\,\underline{|M-2}$$
$$+S(X_1^4X_2^2X_3^2)(N-2)(12N^4-48N^3+192N^2-480N+420)\,\underline{|M-3}$$
$$+S(X_1^4X_2^3X_3)(N-2)(-24N^3+168N^2-360N+280)\,\underline{|M-3}$$
$$+S(X_1^4X_2^2X_3X_4)(N-2)(N-3)(-24N^3+216N^2-840N+840)\,\underline{|M-4}$$
$$+S(X_1^4X_2X_3X_4X_5)(N-2)(N-3)(N-4)(144N^2-1440N+1680)\,\underline{|M-5}$$
$$+S(X_1^3X_2^3X_3^2)(N-2)(-48N^3+240N^2-560N+560)\,\underline{|M-3}$$
$$+S(X_1^3X_2^2X_3^2X_4)(N-2)(N-3)(-48N^3+336N^2-1200N+1680)\,\underline{|M-4}$$
$$+S(X_1^3X_2^3X_3X_4)(N-2)(N-3)(288N^2-960N+1120)\,\underline{|M-4}$$
$$+S(X_1^3X_2^2X_3X_4X_5)(N-2)(N-3)(N-4)(288N^2-1920N+3360)\,\underline{|M-5}$$
$$+S(X_1^3X_2X_3X_4X_5X_6)(N-2)(N-3)(N-4)(N-5)(-2880N+6720)\,\underline{|M-6}$$
$$+S(X_1^2X_2^2X_3^2X_4^2)(N-2)(N-3)(24N^4-96N^3+432N^2-1440N+2520)\,\underline{|M-4}$$
$$+S(X_1^2X_2^2X_3^2X_4X_5)(N-2)(N-3)(N-4)(-48N^3+432N^2-2160N+5040)\,\underline{|M-5}$$
$$+S(X_1^2X_2^2X_3X_4X_5X_6)(N-2)(N-3)(N-4)(N-5)(288N^2-2880N+10080)\,\underline{|M-6}$$
$$+S(X_1^2X_2X_3X_4X_5X_6X_7)(N-2)(N-3)(N-4)(N-5)(N-6)(-2880N+20160)\,\underline{|M-7}$$
$$+S(X_1X_2X_3X_4X_5X_6X_7X_8)(N-2)(N-3)(N-4)(N-5)(N-6)(N-7).40320\,\underline{|M-8}\}.$$

Now the origin being the mean of the sampled population, we have as before $S(X_1)=0$, and thus we have the equations:

$$\begin{aligned}
M\bar{\nu}_8 &= S(X_1^8),\\
0 &= S(X_1^7)\,S(X_1),\\
M^2\bar{\nu}_6\bar{\nu}_2 &= S(X_1^6)\,S(X_1^2),\\
0 &= S(X_1^6)\,\{S(X_1)\}^2,\\
M^2\bar{\nu}_5\bar{\nu}_3 &= S(X_1^5)\,S(X_1^3),\\
0 &= S(X_1^5)\,S(X_1^2)\,S(X_1),\\
0 &= S(X_1^5)\,\{S(X_1)\}^3,\\
M^2\bar{\nu}_4^2 &= \{S(X_1^4)\}^2,\\
0 &= S(X_1^4)\,S(X_1^3)\,S(X_1),\\
M^3\bar{\nu}_4\bar{\nu}_2^2 &= S(X_1^4)\,\{S(X_1^2)\}^2,\\
0 &= S(X_1^4)\,\{S(X_1)\}^4,\\
0 &= S(X_1^4)\,S(X_1^2)\,\{S(X_1)\}^2.
\end{aligned}$$

$$
\begin{aligned}
M^3\bar{\nu}_3^2\bar{\nu}_2 &= \{S(X_1^3)\}^2 S(X_1^2),\\
0 &= \{S(X_1^3)\}^2 \{S(X_1)\}^2,\\
0 &= S(X_1^3)\{S(X_1^2)\}^2 S(X_1),\\
0 &= S(X_1^3) S(X_1^2)\{S(X_1)\}^3,\\
0 &= S(X_1^3)\{S(X_1)\}^5,\\
M^4\bar{\nu}_2^4 &= \{S(X_1^2)\}^4,\\
0 &= \{S(X_1^2)\}^3\{S(X_1)\}^2,\\
0 &= \{S(X_1^2)\}^2\{S(X_1)\}^4,\\
0 &= S(X_1^2)\{S(X_1)\}^6,\\
0 &= \{S(X_1)\}^8.
\end{aligned}
$$

Expanding the right-hand sides of these equations by the appropriate expansions, we obtain 22 linear equations to express the 22 sums $S(X_1^8)$, $S(X_1^7X_2)$, $S(X_1^6X_2^2)$, $S(X_1^6X_2X_3)$, $S(X_1^5X_2^3)$, $S(X_1^5X_2^2X_3)$, $S(X_1^5X_2X_3X_4)$, $S(X_1^4X_2^4)$, $S(X_1^4X_2^2X_3^2)$, $S(X_1^4X_2^3X_3)$, $S(X_1^4X_2^2X_3X_4)$, $S(X_1^4X_2X_3X_4X_5)$, $S(X_1^3X_2^3X_3^2)$, $S(X_1^3X_2^2X_3^2X_4)$, $S(X_1^3X_2^3X_3X_4)$, $S(X_1^3X_2^2X_3X_4X_5)$, $S(X_1^3X_2X_3X_4X_5X_6)$, $S(X_1^2X_2^2X_3^2X_4^2)$, $S(X_1^2X_2^2X_3^2X_4X_5)$, $S(X_1^2X_2^2X_3X_4X_5X_6)$, $S(X_1^2X_2X_3X_4X_5X_6X_7)$, and $S(X_1X_2X_3X_4X_5X_6X_7X_8)$ in terms of the moments $\bar{\nu}_8$, $\bar{\nu}_6\bar{\nu}_2$, $\bar{\nu}_5\bar{\nu}_3$, $\bar{\nu}_4^2$, $\bar{\nu}_4\bar{\nu}_2^2$, $\bar{\nu}_3^2\bar{\nu}_2$ and $\bar{\nu}_2^4$ of the sampled population.

Solving these equations we obtain, after considerable manipulation:

$$
\begin{aligned}
S(X_1^8) &= M\bar{\nu}_8,\\
S(X_1^7X_2) &= -M\bar{\nu}_8,\\
S(X_1^6X_2^2) &= M^2\bar{\nu}_6\bar{\nu}_2 - M\bar{\nu}_8,\\
2S(X_1^6X_2X_3) &= 2M\bar{\nu}_8 - M^2\bar{\nu}_6\bar{\nu}_2,\\
S(X_1^5X_2^3) &= M^2\bar{\nu}_5\bar{\nu}_3 - M\bar{\nu}_8,\\
S(X_1^5X_2^2X_3) &= 2M\bar{\nu}_8 - M^2\bar{\nu}_6\bar{\nu}_2 - M^2\bar{\nu}_5\bar{\nu}_3,\\
6S(X_1^5X_2X_3X_4) &= 3M^2\bar{\nu}_6\bar{\nu}_2 + 2M^2\bar{\nu}_5\bar{\nu}_3 - 6M\bar{\nu}_8,\\
2S(X_1^4X_2^4) &= M^2\bar{\nu}_4^2 - M\bar{\nu}_8,\\
2S(X_1^4X_2^2X_3^2) &= 2M\bar{\nu}_8 - 2M^2\bar{\nu}_6\bar{\nu}_2 - M^2\bar{\nu}_4^2 + M^3\bar{\nu}_4\bar{\nu}_2^2,\\
S(X_1^4X_2^3X_3) &= 2M\bar{\nu}_8 - M^2\bar{\nu}_4^2 - M^2\bar{\nu}_5\bar{\nu}_3,\\
2S(X_1^4X_2^2X_3X_4) &= 3M^2\bar{\nu}_6\bar{\nu}_2 + 2M^2\bar{\nu}_5\bar{\nu}_3 + 2M^2\bar{\nu}_4^2 - M^3\bar{\nu}_4\bar{\nu}_2^2 - 6M\bar{\nu}_8,\\
24S(X_1^4X_2X_3X_4X_5) &= 24M\bar{\nu}_8 - 12M^2\bar{\nu}_6\bar{\nu}_2 - 8M^2\bar{\nu}_5\bar{\nu}_3 - 6M^2\bar{\nu}_4^2 + 3M^3\bar{\nu}_4\bar{\nu}_2^2,\\
2S(X_1^3X_2^3X_3^2) &= 2M\bar{\nu}_8 - M^2\bar{\nu}_6\bar{\nu}_2 - 2M^2\bar{\nu}_5\bar{\nu}_3 + M^3\bar{\nu}_3^2\bar{\nu}_2,\\
2S(X_1^3X_2^2X_3^2X_4) &= 4M^2\bar{\nu}_6\bar{\nu}_2 + 4M^2\bar{\nu}_5\bar{\nu}_3 + M^2\bar{\nu}_4^2 - M^3\bar{\nu}_4\bar{\nu}_2^2\\
&\qquad - 2M^3\bar{\nu}_3^2\bar{\nu}_2 - 6M\bar{\nu}_8,\\
4S(X_1^3X_2^3X_3X_4) &= M^2\bar{\nu}_6\bar{\nu}_2 + 4M^2\bar{\nu}_5\bar{\nu}_3 + 2M^2\bar{\nu}_4^2 - M^3\bar{\nu}_3^2\bar{\nu}_2 - 6M\bar{\nu}_8,\\
6S(X_1^3X_2^2X_3X_4X_5) &= 24M\bar{\nu}_8 - 12M^2\bar{\nu}_6\bar{\nu}_2 - 14M^2\bar{\nu}_5\bar{\nu}_3 - 6M^2\bar{\nu}_4^2\\
&\qquad + 3M^3\bar{\nu}_4\bar{\nu}_2^2 + 5M^3\bar{\nu}_3^2\bar{\nu}_2,
\end{aligned}
$$

$$120S(X_1^3X_2X_3X_4X_5X_6) = 120M\bar{\nu}_8 + 60M^2\bar{\nu}_6\bar{\nu}_2 + 64M^2\bar{\nu}_5\bar{\nu}_3 + 30M^2\bar{\nu}_4^2 - 15M^3\bar{\nu}_4\bar{\nu}_2^2 - 20M^3\bar{\nu}_3^2\bar{\nu}_2$$

$$24S(X_1^2X_2^2X_3^2X_4^2) = 8M^2\bar{\nu}_6\bar{\nu}_2 + 3M^2\bar{\nu}_4^2 - 6M^3\bar{\nu}_4\bar{\nu}_2^2 + M^4\bar{\nu}_2^4 - 6M\bar{\nu}_8,$$

$$12S(X_1^2X_2^2X_3^2X_4X_5) = 24M\bar{\nu}_8 - 20M^2\bar{\nu}_6\bar{\nu}_2 - 12M^2\bar{\nu}_5\bar{\nu}_3 - 6M^2\bar{\nu}_4^2 + 9M^3\bar{\nu}_4\bar{\nu}_2^2 + 6M^3\bar{\nu}_3^2\bar{\nu}_2 - M^4\bar{\nu}_2^4,$$

$$48S(X_1^3X_2^2X_3X_4X_5X_6) = 84M^2\bar{\nu}_6\bar{\nu}_2 + 64M^2\bar{\nu}_5\bar{\nu}_3 + 30M^2\bar{\nu}_4^2 - 33M^3\bar{\nu}_4\bar{\nu}_2^2 - 28M^3\bar{\nu}_3^2\bar{\nu}_2 + 3M^4\bar{\nu}_2^4 - 120M\bar{\nu}_8,$$

$$720S(X_1^2X_2X_3X_4X_5X_6X_7) = 180M^3\bar{\nu}_4\bar{\nu}_2^2 + 160M^3\bar{\nu}_3^2\bar{\nu}_2 - 15M^4\bar{\nu}_2^4 - 180M^2\bar{\nu}_4^2 - 384M^2\bar{\nu}_5\bar{\nu}_3 - 480M^2\bar{\nu}_6\bar{\nu}_2 - 720M\bar{\nu}_8,$$

$$40320S(X_1X_2X_3X_4X_5X_6X_7X_8) = 21840M\bar{\nu}_8 + 3360M^2\bar{\nu}_6\bar{\nu}_2 + 2688M^2\bar{\nu}_5\bar{\nu}_3 + 1260M^3\bar{\nu}_4^2 - 1260M^3\bar{\nu}_4\bar{\nu}_2^2 - 1120M^3\bar{\nu}_3^2\bar{\nu}_2 + 105M^4\bar{\nu}_2^4.$$

Substituting in the last expression for ${}_2M_4'$, we have

$${}_2M_4' = \frac{N-1}{N^7}\Big\{(N-1)^3\bar{\nu}_8 + \frac{(8N^3-24N^2+24N-8)}{M-1}\bar{\nu}_8$$

$$+ \frac{(4N^4-16N^3+48N^2-64N+28)}{M-1}(M\bar{\nu}_6\bar{\nu}_2 - \bar{\nu}_8)$$

$$+ \frac{(N-2)(-4N^3+36N^2-60N+28)}{(M-1)(M-2)}(2\bar{\nu}_8 - M\bar{\nu}_6\bar{\nu}_2)$$

$$+ \frac{(-24N^3+72N^2-104N+56)}{M-1}(M\bar{\nu}_5\bar{\nu}_3 - \bar{\nu}_8)$$

$$+ \frac{(N-2)(-24N^3+120N^2-264N+168)}{(M-1)(M-2)}(2\bar{\nu}_8 - M\bar{\nu}_6\bar{\nu}_2 - M\bar{\nu}_5\bar{\nu}_3)$$

$$+ \frac{(N-2)(N-3)(24N^2-80N+56)}{(M-1)(M-2)(M-3)}(3M\bar{\nu}_6\bar{\nu}_2 + 2M\bar{\nu}_5\bar{\nu}_3 - 6\bar{\nu}_8)$$

$$+ \frac{(3N^4-12N^3+42N^2-60N+35)}{M-1}(M\bar{\nu}_4^2 - \bar{\nu}_8)$$

$$+ \frac{(N-2)(6N^4-24N^3+96N^2-240N+210)}{(M-1)(M-2)}(2\bar{\nu}_8 - 2M\bar{\nu}_6\bar{\nu}_2 - M\bar{\nu}_4^2 + M^2\bar{\nu}_4\bar{\nu}_2^2)$$

$$+ \frac{(N-2)(-24N^3+168N^2-360N+280)}{(M-1)(M-2)}(2\bar{\nu}_8 - M\bar{\nu}_4^2 - M\bar{\nu}_5\bar{\nu}_3)$$

$$+ \frac{(N-2)(N-3)(-12N^3+108N^2-420N+420)}{(M-1)(M-2)(M-3)}(3M\bar{\nu}_6\bar{\nu}_2 + 2M\bar{\nu}_5\bar{\nu}_3 + 2M\bar{\nu}_4^2 - M^2\bar{\nu}_4\bar{\nu}_2^2 - 6\bar{\nu}_8)$$

$$+ \frac{(N-2)(N-3)(N-4)(6N^2-60N+70)}{(M-1)(M-2)(M-3)(M-4)}(24\bar{\nu}_8 - 12M\bar{\nu}_6\bar{\nu}_2 - 8M\bar{\nu}_5\bar{\nu}_3 - 6M\bar{\nu}_4^2 + 3M^2\bar{\nu}_4\bar{\nu}_2^2)$$

$$+ \frac{(N-2)(-24N^3+120N^2-280N+280)}{(M-1)(M-2)}(2\bar{\nu}_8 + M^2\bar{\nu}_3^2\bar{\nu}_2 - M\bar{\nu}_6\bar{\nu}_2 - 2M\bar{\nu}_5\bar{\nu}_3)$$

$$+ \frac{(N-2)(N-3)(-24N^3+168N^2-600N+840)}{(M-1)(M-2)(M-3)}(4M\bar{\nu}_6\bar{\nu}_2 + 4M\bar{\nu}_5\bar{\nu}_3 + M\bar{\nu}_4^2 - M^2\bar{\nu}_4\bar{\nu}_2^2 - 2M^2\bar{\nu}_3^2\bar{\nu}_2 - 6\bar{\nu}_8)$$

$$+\frac{(N-2)(N-3)(72N^2-240N+280)}{(M-1)(M-2)(M-3)}(M\bar{\nu}_6\bar{\nu}_2+4M\bar{\nu}_5\bar{\nu}_3+2M\bar{\nu}_4^2 - M^2\bar{\nu}_3^2\bar{\nu}_2-6\bar{\nu}_8)$$

$$+\frac{(N-2)(N-3)(N-4)(48N^2-320N+560)}{(M-1)(M-2)(M-3)(M-4)}(24\bar{\nu}_8-12M\bar{\nu}_6\bar{\nu}_2 - 14M\bar{\nu}_5\bar{\nu}_3-6M\bar{\nu}_4^2+3M^2\bar{\nu}_4\bar{\nu}_2^2+5M^2\bar{\nu}_3^2\bar{\nu}_2)$$

$$+\frac{(N-2)(N-3)(N-4)(N-5)(-24N+56)}{(M-1)(M-2)(M-3)(M-4)(M-5)}(120\bar{\nu}_8+60M\bar{\nu}_6\bar{\nu}_2 + 64M\bar{\nu}_5\bar{\nu}_3+30M\bar{\nu}_4^2-15M^2\bar{\nu}_4\bar{\nu}_2^2-20M^2\bar{\nu}_3^2\bar{\nu}_2)$$

$$+\frac{(N-2)(N-3)(N^4-4N^3+18N^2-60N+105)}{(M-1)(M-2)(M-3)}(8M\bar{\nu}_6\bar{\nu}_2+3M\bar{\nu}_4^2 - 6M^2\bar{\nu}_4\bar{\nu}_2^2+M^3\bar{\nu}_2^4-6\bar{\nu}_8)$$

$$+\frac{(N-2)(N-3)(N-4)(-4N^3+36N^2-180N+420)}{(M-1)(M-2)(M-3)(M-4)}(24\bar{\nu}_8-20M\bar{\nu}_6\bar{\nu}_2 - 12M\bar{\nu}_5\bar{\nu}_3-6M\bar{\nu}_4^2+9M^2\bar{\nu}_4\bar{\nu}_2^2+6M^2\bar{\nu}_3^2\bar{\nu}_2-M^3\bar{\nu}_2^4)$$

$$+\frac{(N-2)(N-3)(N-4)(N-5)(6N^2-60N+210)}{(M-1)(M-2)(M-3)(M-4)(M-5)}(84M\bar{\nu}_6\bar{\nu}_2 + 64M\bar{\nu}_5\bar{\nu}_3+30M\bar{\nu}_4^2-33M^2\bar{\nu}_4\bar{\nu}_2^2-28M^2\bar{\nu}_3^2\bar{\nu}_2+3M^3\bar{\nu}_2^4-120\bar{\nu}_8)$$

$$+\frac{(N-2)(N-3)(N-4)(N-5)(N-6)(-4N+28)}{(M-1)(M-2)(M-3)(M-4)(M-5)(M-6)}(180M^2\bar{\nu}_4\bar{\nu}_2^2 + 160M^2\bar{\nu}_3^2\bar{\nu}_2-15M^2\bar{\nu}_2^4-180M\bar{\nu}_4^2-384M\bar{\nu}_5\bar{\nu}_3-480M\bar{\nu}_6\bar{\nu}_2-720\bar{\nu}_8)$$

$$+\frac{(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-1)(M-2)(M-3)(M-4)(M-5)(M-6)(M-7)}(21840\bar{\nu}_8+3360M\bar{\nu}_6\bar{\nu}_2+2688M\bar{\nu}_5\bar{\nu}_3+1260M\bar{\nu}_4^2-1260M^2\bar{\nu}_4\bar{\nu}_2^2 - 1120M^2\bar{\nu}_3^2\bar{\nu}_2+105M^3\bar{\nu}_2^4)\Big\},$$

and hence

$${}_2M_4'=\frac{N-1}{N^7}\Big\{\bar{\nu}_8\Big[(N-1)^3-\frac{7N^4-60N^3+186N^2-252N+127}{M-1} + \frac{4(N-2)(3N^4-50N^3+270N^2-602N+483)}{(M-1)(M-2)} - \frac{6(N-2)(N-3)(N^4-40N^3+390N^2-1400N+1701)}{(M-1)(M-2)(M-3)} - \frac{48(N-2)(N-3)(N-4)(2N^3-45N^2+280N-525)}{(M-1)(M-2)(M-3)(M-4)} - \frac{240(N-2)(N-3)(N-4)(N-5)(3N^2-18N+77)}{(M-1)(M-2)(M-3)(M-4)(M-5)} + \frac{2880(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-1)(M-2)(M-3)(M-4)(M-5)(M-6)} + \frac{21840(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-1)(M-2)(M-3)(M-4)(M-5)(M-6)(M-7)}\Big]$$

$$+\frac{4M\bar{\nu}_6\bar{\nu}_2}{M-1}\Bigg[(N-1)^2(\overline{N-1}^2+6)-\frac{(N-2)(3N^4-25N^3+117N^2-271N+224)}{M-2}$$
$$+\frac{(N-2)(N-3)(2N^4-41N^3+321N^2-1155N+1477)}{(M-2)(M-3)}$$
$$+\frac{2(N-2)(N-3)(N-4)(10N^3-171N^2+1020N-1995)}{(M-2)(M-3)(M-4)}$$
$$+\frac{6(N-2)(N-3)(N-4)(N-5)(21N^2-270N+875)}{(M-2)(M-3)(M-4)(M-5)}$$
$$+\frac{480(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-2)(M-3)(M-4)(M-5)(M-6)}$$
$$+\frac{840(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-2)(M-3)(M-4)(M-5)(M-6)(M-7)}\Bigg]$$

$$-\frac{8M\bar{\nu}_5\bar{\nu}_3}{M-1}\Bigg[(N-1)(3\overline{N-1}^2+4)-\frac{2(N-2)(6N^3-33N^2+74N-63)}{M-2}$$
$$+\frac{(N-2)(N-3)(15N^3-153N^2+545N-679)}{(M-2)(M-3)}$$
$$-\frac{(N-2)(N-3)(N-4)(6N^3-144N^2+890N-1680)}{(M-2)(M-3)(M-4)}$$
$$-\frac{16(N-2)(N-3)(N-4)(N-5)(3N^2-42N+133)}{(M-2)(M-3)(M-4)(M-5)}$$
$$-\frac{192(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-2)(M-3)(M-4)(M-5)(M-6)}$$
$$-\frac{336(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-2)(M-3)(M-4)(M-5)(M-6)(M-7)}\Bigg]$$

$$+\frac{M\bar{\nu}_4^2}{M-1}\Bigg[3(N-1)^4+24(N-1)^2+8$$
$$-\frac{2(N-2)(3N^4-24N^3+132N^2-300N+245)}{M-2}$$
$$+\frac{(N-2)(N-3)(3N^4-60N^3+582N^2-2100N+2555)}{(M-2)(M-3)}$$
$$+\frac{12(N-2)(N-3)(N-4)(2N^3-45N^2+280N-525)}{(M-2)(M-3)(M-4)}$$
$$+\frac{60(N-2)(N-3)(N-4)(N-5)(3N^2-42N+133)}{(M-2)(M-3)(M-4)(M-5)}$$
$$+\frac{720(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-2)(M-3)(M-4)(M-5)(M-6)}$$
$$+\frac{1260(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-2)(M-3)(M-4)(M-5)(M-6)(M-7)}\Bigg]$$

$$+\frac{6M^2(N-2)\bar{\nu}_4\bar{\nu}_2^2}{(M-1)(M-2)}\Big[(N-1)^4+10(N-1)^2-16(N-1)+8$$
$$-\frac{(N-3)(N^4-10N^3+64N^2-230N+315)}{M-3}$$
$$-\frac{(N-3)(N-4)(6N^3-81N^2+460N-945)}{(M-3)(M-4)}$$
$$-\frac{(N-3)(N-4)(N-5)(33N^2-390N+1295)}{(M-3)(M-4)(M-5)}$$
$$-\frac{120(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-3)(M-4)(M-5)(M-6)}$$
$$-\frac{210(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-3)(M-4)(M-5)(M-6)(M-7)}\Big]$$

$$-\frac{8M^2(N-2)\bar{\nu}_3^2\bar{\nu}_2}{(M-1)(M-2)}\Big[3(N-1)^3-6(N-1)^2+14(N-1)-12$$
$$-\frac{(N-3)(6N^3-51N^2+180N-245)}{M-3}$$
$$+\frac{(N-3)(N-4)(3N^3-57N^2+335N-665)}{(M-3)(M-4)}$$
$$+\frac{(N-3)(N-4)(N-5)(21N^2-270N+875)}{(M-3)(M-4)(M-5)}$$
$$+\frac{80(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-3)(M-4)(M-5)(M-6)}$$
$$+\frac{140(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-3)(M-4)(M-5)(M-6)(M-7)}\Big]$$

$$+\frac{M^3(N-2)(N-3)\bar{\nu}_2^4}{(M-1)(M-2)(M-3)}\Big[(N-1)^4+12(N-1)^2-32(N-1)+60$$
$$+\frac{4(N-4)(N^3-9N^2+45N-105)}{M-4}$$
$$+\frac{9(N-4)(N-5)(2N^2-20N+70)}{(M-4)(M-5)}$$
$$+\frac{60(N-4)(N-5)(N-6)(N-7)}{(M-4)(M-5)(M-6)}$$
$$+\frac{105(N-4)(N-5)(N-6)(N-7)}{(M-4)(M-5)(M-6)(M-7)}\Big]\Big\}.$$

Transferring to the mean of this distribution of σ^2 of samples as origin, by

$$_2M_4={}_2M_4'-4\,.\,{}_2M_3\,.\,{}_2M_1'-6\,.\,{}_2M_2\,.\,({}_2M_1')^2-({}_2M_1')^4,$$

we have

$$
\begin{aligned}
{}_2M_4 = \frac{N-1}{N^7}\Bigg\{\bar{\nu}_8\Bigg[(N-1)^3 &- \frac{7N^4-60N^3+186N^2-252N+127}{M-1} \\
&+ \frac{4(N-2)(3N^4-50N^3+270N^2-602N+483)}{(M-1)(M-2)} \\
&- \frac{6(N-2)(N-3)(N^4-40N^3+390N^2-1400N+1701)}{(M-1)(M-2)(M-3)} \\
&- \frac{48(N-2)(N-3)(N-4)(2N^3-45N^2+280N-525)}{(M-1)(M-2)(M-3)(M-4)} \\
&- \frac{240(N-2)(N-3)(N-4)(N-5)(3N^2-18N+77)}{(M-1)(M-2)(M-3)(M-4)(M-5)} \\
&+ \frac{2880(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-1)(M-2)(M-3)(M-4)(M-5)(M-6)} \\
&+ \frac{21840(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-1)(M-2)(M-3)(M-4)(M-5)(M-6)(M-7)}\Bigg] \\
- \frac{4M\bar{\nu}_6\bar{\nu}_2}{M-1}\Bigg[(N-1)^2(N-7) &- \frac{N(N-1)(3N^3-21N^2+45N-31)}{M-1} \\
&+ \frac{(N-2)(3N^4-25N^3+117N^2-217N+224)}{M-2} \\
&+ \frac{2N(N-1)(N-2)(N-3)(N^2-15N+30)}{(M-1)(M-2)} \\
&- \frac{(N-2)(N-3)(2N^4-41N^3+321N^2-1155N+1477)}{(M-2)(M-3)} \\
&+ \frac{6N(N-1)(N-2)(N-3)(3N^2-30N+65)}{(M-1)(M-2)(M-3)} \\
&- \frac{2(N-2)(N-3)(N-4)(10N^3-171N^2+1020N-1995)}{(M-2)(M-3)(M-4)} \\
&+ \frac{72N(N-1)(N-2)(N-3)(N-4)(N-5)}{(M-1)(M-2)(M-3)(M-4)} \\
&- \frac{6(N-2)(N-3)(N-4)(N-5)(21N^2-270N+875)}{(M-2)(M-3)(M-4)(M-5)} \\
&+ \frac{120N(N-1)(N-2)(N-3)(N-4)(N-5)}{(M-1)(M-2)(M-3)(M-4)(M-5)} \\
&- \frac{480(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-2)(M-3)(M-4)(M-5)(M-6)} \\
&- \frac{840(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-2)(M-3)(M-4)(M-5)(M-6)(M-7)}\Bigg]
\end{aligned}
$$

$$-\frac{8M\bar{\nu}_3\bar{\nu}_2}{M-1}\Bigg[(N-1)(3\overline{N-1}^2+4)-\frac{2(N-2)(6N^3-33N^2+74N-63)}{M-2}$$
$$+\frac{(N-2)(N-3)(15N^3-153N^2+545N-679)}{(M-2)(M-3)}$$
$$-\frac{(N-2)(N-3)(N-4)(6N^3-144N^2+890N-1680)}{(M-2)(M-3)(M-4)}$$
$$-\frac{16(N-2)(N-3)(N-4)(N-5)(3N^2-42N+133)}{(M-2)(M-3)(M-4)(M-5)}$$
$$-\frac{192(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-2)(M-3)(M-4)(M-5)(M-6)}$$
$$-\frac{336(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-2)(M-3)(M-4)(M-5)(M-6)(M-7)}\Bigg]$$
$$+\frac{M\bar{\nu}_4^2}{M-1}\Bigg[3(N-1)^4+24(N-1)^2+8$$
$$-\frac{2(N-2)(3N^4-24N^3+132N^2-300N+245)}{M-2}$$
$$+\frac{(N-2)(N-3)(3N^4-60N^3+582N^2-2100N+2555)}{(M-2)(M-3)}$$
$$+\frac{12(N-2)(N-3)(N-4)(2N^3-45N^2+280N-525)}{(M-2)(M-3)(M-4)}$$
$$+\frac{60(N-2)(N-3)(N-4)(N-5)(3N^2-42N+133)}{(M-2)(M-3)(M-4)(M-5)}$$
$$+\frac{720(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-2)(M-3)(M-4)(M-5)(M-6)}$$
$$+\frac{1260(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-2)(M-3)(M-4)(M-5)(M-6)(M-7)}\Bigg]$$
$$-\frac{6M^2\bar{\nu}_4\bar{\nu}_2^2}{M-1}\Bigg[\frac{N(N-1)^2(N^2-3N+10)}{M-1}$$
$$-\frac{(N-2)(N^4-4N^3+16N^2-40N+35)}{M-2}$$
$$+\frac{N^2(N-1)^2(N^2-6N+7)}{(M-1)^2}$$
$$-\frac{2N(N-1)(N-2)(N^3-8N^2+31N-40)}{(M-1)(M-2)}$$
$$+\frac{(N-2)(N-3)(N^4-10N^3+64N^2-230N+315)}{(M-2)(M-3)}$$
$$+\frac{4N^2(N-1)^2(N-2)(N-3)}{(M-1)^2(M-2)}$$

$$
\begin{aligned}
&- \frac{2N(N-1)(N-2)(N-3)(5N^2-42N+95)}{(M-1)(M-2)(M-3)} \\
&+ \frac{(N-2)(N-3)(N-4)(6N^3-81N^2+460N-945)}{(M-2)(M-3)(M-4)} \\
&+ \frac{6N^2(N-1)^2(N-2)(N-3)}{(M-1)^2(M-2)(M-3)} \\
&- \frac{36N(N-1)(N-2)(N-3)(N-4)(N-5)}{(M-1)(M-2)(M-3)(M-4)} \\
&+ \frac{(N-2)(N-3)(N-4)(N-5)(33N^2-390N+1295)}{(M-2)(M-3)(M-4)(M-5)} \\
&- \frac{60N(N-1)(N-2)(N-3)(N-4)(N-5)}{(M-1)(M-2)(M-3)(M-4)(M-5)} \\
&+ \frac{120(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-2)(M-3)(M-4)(M-5)(M-6)} \\
&+ \frac{210(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-2)(M-3)(M-4)(M-5)(M-6)(M-7)}\Big] \\
+ \frac{8M^2\bar{\nu}_2^2\bar{\nu}_2}{M-1}\Big[&\frac{N(N-1)(3N^2-6N+5)}{M-1} \\
&- \frac{(N-2)(3N^3-15N^2+35N-35)}{M-2} \\
&- \frac{6N(N-1)(N-2)(N^2-4N+5)}{(M-1)(M-2)} \\
&+ \frac{(N-2)(N-3)(6N^3-51N^2+180N-245)}{(M-2)(M-3)} \\
&+ \frac{N(N-1)(N-2)(N-3)(3N^2-30N+65)}{(M-1)(M-2)(M-3)} \\
&- \frac{(N-2)(N-3)(N-4)(3N^3-57N^2+335N-665)}{(M-2)(M-3)(M-4)} \\
&+ \frac{12N(N-1)(N-2)(N-3)(N-4)(N-5)}{(M-1)(M-2)(M-3)(M-4)} \\
&- \frac{(N-2)(N-3)(N-4)(N-5)(21N^2-270N+875)}{(M-2)(M-3)(M-4)(M-5)} \\
&+ \frac{20N(N-1)(N-2)(N-3)(N-4)(N-5)}{(M-1)(M-2)(M-3)(M-4)(M-5)} \\
&- \frac{80(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-2)(M-3)(M-4)(M-5)(M-6)} \\
&- \frac{140(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-2)(M-3)(M-4)(M-5)(M-6)(M-7)}\Big]
\end{aligned}
$$

$$+\frac{M^3\bar{\nu}_2^4}{M-1}\left[\frac{3N^2(N-1)^2(N^2-3N+6)}{(M-1)^2}\right.$$
$$-\frac{4N(N-1)(N-2)(N^3-3N^2+9N-15)}{(M-1)(M-2)}$$
$$+\frac{(N-2)(N-3)(N^4-4N^3+18N^2-60N+105)}{(M-2)(M-3)}$$
$$-\frac{3N^3(N-1)^3}{(M-1)^3}+\frac{12N^2(N-1)^2(N-2)(N-3)}{(M-1)^2(M-2)}$$
$$-\frac{12N(N-1)(N-2)(N-3)(N^2-6N+15)}{(M-1)(M-2)(M-3)}$$
$$+\frac{4(N-2)(N-3)(N-4)(N^3-9N^2+45N-105)}{(M-2)(M-3)(M-4)}$$
$$+\frac{18N^2(N-1)^2(N-2)(N-3)}{(M-1)^2(M-2)(M-3)}$$
$$-\frac{36N(N-1)(N-2)(N-3)(N-4)(N-5)}{(M-1)(M-2)(M-3)(M-4)}$$
$$+\frac{9(N-2)(N-3)(N-4)(N-5)(2N^2-20N+70)}{(M-2)(M-3)(M-4)(M-5)}$$
$$-\frac{60N(N-1)(N-2)(N-3)(N-4)(N-5)}{(M-1)(M-2)(M-3)(M-4)(M-5)}$$
$$+\frac{60(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-2)(M-3)(M-4)(M-5)(M-6)}$$
$$\left.+\frac{105(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}{(M-2)(M-3)(M-4)(M-5)(M-6)(M-7)}\right]\Big\}$$
.....(4).

The Tchouproff formulae (1), (2), (3) and (4) of Part III are particular cases of formulae (1), (2), (3) and (4) of this part and are deduced by putting M infinite.

The great length of these formulae (3) and (4) is of interest from several points of view. The method employed to obtain them can be employed to obtain the higher moments ${}_2M_5$, ${}_2M_6$, etc. of the Distribution of σ^2 of Samples, and, indeed, any moment of a distribution of a higher moment of the sample, for instance the moments of the distribution of the fourth moments of samples. It is however immediately apparent when the problem is considered, that the algebra involved would be extremely laborious and the results obtained of too great length to be of any value. It is thus evident that the expression of the moments of various distributions, of the nature considered here, in terms of the moments (or betas) of the sampled population, although quite possible, leads to unduly complex results and some alternative method of expression is required.

To return to ${}_2M_1$, ${}_2M_2$, ${}_2M_3$ and ${}_2M_4$, the size of the last two formulae, together with the type of term involved, means that ${}_2M_4$ is in practice impossible of calculation, even with a machine, when M is over 40 or 50; consequently any sampled population of fair size must be treated as infinite and Tchouproff's formulae used even though the assumption be not really legitimate and thus the results

obtained only approximate. The labour involved, even when this assumption is made, is moreover still very considerable.

Practical considerations thus compel us, when a representation of the Distribution of σ^2 of Samples by a Pearson Curve is required, to use Tchouproff's formulae when the population is anything except very small. When it is very small however the formulae given above may be used and a Pearson Curve obtained to represent the distribution. The possibility of such a representation will now be considered.

The Distribution of σ^2 of Samples of 10 *from Population* (C) *of total frequency* 36, *together with its representation by a Pearson Curve.*

Now the types of samples to be obtained from (C), together with the σ^2 of each type of sample, are:

Type of Sample	Frequency	σ^2
0 *A* 0 *B* 10 *C*	3,268,760	·00
0 *A* 1 *B* 9 *C*	20,429,750	·09
0 *A* 2 *B* 8 *C*	48,670,875	·16
0 *A* 3 *B* 7 *C*	57,684,000	·21
0 *A* 4 *B* 6 *C*	37,191,000	·24
0 *A* 5 *B* 5 *C*	13,388,760	·25
0 *A* 6 *B* 4 *C*	2,656,500	·24
0 *A* 7 *B* 3 *C*	276,000	·21
0 *A* 8 *B* 2 *C*	13,500	·16
0 *A* 9 *B* 1 *C*	250	·09
0 *A* 10 *B* 0 *C*	1	·00
1 *A* 0 *B* 9 *C*	2,042,975	·36
1 *A* 1 *B* 8 *C*	10,815,750	·41
1 *A* 2 *B* 7 *C*	21,631,500	·44
1 *A* 3 *B* 6 *C*	21,252,000	·45
1 *A* 4 *B* 5 *C*	11,157,300	·44
1 *A* 5 *B* 4 *C*	3,187,800	·41
1 *A* 6 *B* 3 *C*	483,000	·36
1 *A* 7 *B* 2 *C*	36,000	·29
1 *A* 8 *B* 1 *C*	1,125	·20
1 *A* 9 *B* 0 *C*	10	·09

Thus dividing the samples to be obtained into two groups according as they do or do not contain the element of the first category of (C), we have the two distributions of σ^2 of samples as follows:

Group I		Group II	
Value of σ^2	Frequency	Value of σ^2	Frequency
·09	10	*·00*	3,268,761
·20	1,125	*·09*	20,430,000
·29	36,000	*·16*	48,684,375
·36	2,525,975	*·21*	57,960,000
·41	14,003,550	*·24*	39,847,500
·44	32,788,800	*·25*	13,388,760
·45	21,252,000		

Adding these we have for the complete Distribution of σ^2 of Samples of 10 from Population (C):

Value of σ^2	Frequency
·00	3,268,761
·09	20,430,010
·16	48,684,375
·20	1,125
·21	57,960,000
·24	39,847,500
·25	13,388,760
·29	36,000
·36	2,525,975
·41	14,003,550
·44	32,788,800
·45	21,252,000
Total Frequency = 254,186,856	

The constants of this distribution are:

Working origin = ·24. Working unit, the squared unit of (C).

$\nu_1' = \cdot 017,142,86,$ $\quad {}_2M_2 = \cdot 014,274,0,$

$\nu_2' = \cdot 014,567,91,$ $\quad {}_2M_3 = \cdot 000,801,8,$

$\nu_3' = \cdot 001,540,90,$ $\quad {}_2M_4 = \cdot 000,428,2,$

$\nu_4' = \cdot 000,508,45,$

Mean $= 0\cdot 257,143,$ $\quad \Sigma_2 = 0\cdot 119,474,$

${}_2B_1 = 0\cdot 221,053,$ $\quad {}_2B_2 = 2\cdot 101,626.$

A consideration of the two groups of σ^2 of samples above and a comparison of them with the corresponding groups of the Distribution of Means of Samples from (C) given in Part IV illustrate clearly a number of important points that arise in sampling from a finite population.

If the samples be divided into groups by the consideration of the numbers of individuals from certain categories of the sampled population which are to enter into the sample, these groups will lead to a number of sub-frequency distributions of means, or σ^2 of samples as the case may be, which in turn combine into the complete distributions of those constants.

The very nature or definition of a mean ensures that in the case of the distribution of means of samples, these sub-frequency distributions will be moderately skew and uni-modal, whilst the modes of consecutive sub-frequencies will occur at comparatively small intervals of the range of the mean and the ranges of consecutive sub-frequencies will overlap except for small intervals at their ends. Thus the complete Distribution of Means of Samples of N will approximate quite closely to normality even when the number of categories in the sampled population is small and therefore the number of sub-frequencies of means small likewise. In fact for all sampled populations small or large the Distribution of Means of Samples of N, where N is not large, will be strikingly normal in character. This

has been shown in the previous parts of this paper and has been well known before, at least for samples from large populations.

When we turn to the Distribution of σ^2 of Samples of N, where N is not large, matters are quite different. The value of σ^2 measures the dispersion of the individuals of a sample from its mean, thus in any group of samples as defined above there is a tendency for many values of σ^2 to occur twice causing the corresponding sub-frequency of σ^2 of samples to be much more skew than in the case of the means of samples, although it is still uni-modal. However, this also causes the modes and ranges of consecutive sub-frequencies to differ comparatively widely; and thus even when the sampled population is large with many categories the Distribution of σ^2 of Samples is still far from normal, being in general quite distinctly skew. When however the sampled population is very small with but a few categories, these wide differences between the modes and ranges of the sub-frequencies of σ^2 of samples cause the Distribution of σ^2 of Samples itself to be bi- or tri-modal and moreover to be a discrete frequency distribution with unequal intervals between frequencies. As a Pearson Curve is based on the assumption of a uni-modal frequency, it is thus not to be expected that, in this case, a representation by such a curve will be reasonably accurate; accordingly when these formulae do give the moments of the Distribution of σ^2 of Samples with moderate ease and rapidity their practical use is nullified by the failure of any adequate representation.

This failure is clearly shown in the distribution of σ^2 of samples of 10 from (C). Its moment-coefficients are

$$ {}_2M_1' = 0{\cdot}257{,}143, \qquad {}_2M_2 = 0{\cdot}014{,}274, $$
$$ {}_2M_3 = 0{\cdot}000{,}802, \qquad {}_2M_4 = 0{\cdot}000{,}428, $$

and, when the values of the first eight moment-coefficients of (C), together with $M = 36$ and $N = 10$, are substituted in formulae (1), (2), (3) and (4) of this part, the values obtained for ${}_2M_1'$, etc. are precisely the same. Consequently the formulae are quite efficacious, but as the Pearson Curve derived from these moment-coefficient values above proves to be a J-curve, the completeness of the failure is at once indicated.

We can thus sum up the considerations which arise when dealing with the Distribution of σ^2 of Samples from a finite population. Owing to the unwieldy character of the expressions for ${}_2M_3$ and ${}_2M_4$, it is a practical impossibility, even with a calculating machine, to evaluate them both when M is greater than 40 or 50. Thus, when sampling is conducted on the basis of M finite (i.e. complete samples are drawn) and M is of moderate size, we are compelled by sheer necessities of time and labour to represent the resulting Distribution of σ^2 of Samples by a Pearson Curve based on the doubtful assumption that M is infinite, because Tchouproff's formulae, as used in Part III, provide the only reasonable method of calculating the values of ${}_2M_2$, ${}_2M_3$ and ${}_2M_4$. When M is very large, e.g. several thousands, this probably matters little as is indicated by the Distribution II (*b*) of this paper, but when M is only of moderate size, the loss of accuracy in representation is not clearly known and may be considerable; however the assumption appears at present a necessity.

When M is quite small, these four formulae will give the moment-coefficients of the Distribution of σ^2 of Samples moderately easily, but their practical use is liable to cease at this stage. As illustrated here the Distribution of σ^2 of Samples, when M is quite small, may be multi-modal, and moreover is almost certain to be a distribution with discrete frequency groups at unequal intervals.

Taking the first of these difficulties, the plurality of modes, this may at first sight be overcome by developing a set of frequency curves of the Pearson type with more modes than one or by using curves based upon some method such as Thiele's semi-invariants. Now if methods like the latter are to produce multi-modal curves, it appears to me, although I have no practical experience in this field, that more constants than the apparently usual number of four would be required, for these constants are after all equivalent to moments in their essentials. The formula for ${}_2M_4$ is severe enough and it is easily seen that formulae for any other higher moment-coefficients or combinations of such moment-coefficients would be impossibly severe; thus this method would not obviate the difficulty of the evaluation of the necessary constants and would fail in practice at any rate on this account. It might be possible to develop a quasi-Pearson Curve with two or more modes whose constants only involve the first four moment-coefficients of the distribution, but, even if developed, it is a very open question as to whether it would be efficacious in the representation of actual distributions of this type, and until the formulae can be simplified, the use of higher moment-coefficients is not practicable.

The second of the difficulties, that is, the frequency groups becoming discrete and at unequal intervals, presents a problem when the representation of the distribution is required, which is also very difficult and for which, at present, no one appears to have obtained any satisfactory solution.

Enough has been said to emphasise the fact that the problem of the representation of Distributions of σ^2 of Samples from a quite small finite population is one which will call for a large amount of research before any practical advance towards its solution is likely to be made.

In conclusion a comparison is given between the Distribution of σ^2 of Samples of 10 from Population (C) and the corresponding Pearson Curve obtained from the values of ${}_2M_2$, ${}_2M_3$ and ${}_2M_4$ given by the formulae of this part, not because it can be expected that a Pearson Curve will represent the distribution successfully, for a Pearson Curve is designedly uni-modal, but merely to emphasise the unusual character of the distribution when M is very small and also the need for some other method of representation.

For this attempted representation of the Distribution of σ^2 of Samples of 10 from (C) we have seen that the constants of the curve are to be:

$$\text{Mean} = 0{\cdot}257{,}143, \qquad {}_2M_2 = 0{\cdot}014{,}274,$$
$${}_2B_1 = 0{\cdot}221{,}053, \qquad {}_2B_2 = 2{\cdot}101{,}626.$$

The curve is a J-curve of Type I, hence with the usual notation

$$r = 2{\cdot}147{,}820, \qquad \epsilon = 1{\cdot}072{,}312,$$
$$m'^2 - 2{\cdot}147{,}820\, m' + 1{\cdot}072{,}312 = 0.$$

The last equation has roots 1·358,463 and 0·789,356, and as ${}_2M_3$ is positive

$$m_1' = \ 0{\cdot}789{,}356, \qquad m_2' = 1{\cdot}358{,}463;$$

therefore

$$m_1 = -0{\cdot}210{,}644, \qquad m_2 = 0{\cdot}358{,}463.$$

Taking as unit $\frac{1}{100}$ the squared unit of (C)

$$b = 43{\cdot}9659, \qquad a_1 = -62{\cdot}6514, \qquad a_2 = 106{\cdot}6173.$$

Taking the total frequency to be 1000, $y_0 = 29{\cdot}6752$, and the curve is

$$y = 29{\cdot}6752 \left(\frac{x}{62{\cdot}6514} - 1\right)^{-0{\cdot}210{,}644} \left(1 - \frac{x}{106{\cdot}6173}\right)^{0{\cdot}358{,}463}.$$

The Origin = Mode = 25·7143 − 78·8095 = − 53·0952. Hence, in units of Population (C), the range is from $\sigma^2 = {\cdot}096$ to $\sigma^2 = {\cdot}535$.

Keeping to the unit of the curve, the ordinates of the curve, which is shown in Fig. VIII, are

Abscissa	Ordinate	Abscissa	Ordinate
63	64·2940	90	18·1488
66	38·9144	95	15·4086
70	31·7737	100	12·2180
75	27·2467	103	9·6807
80	23·8151	105	7·1806
85	20·8089	106·62	0·0000

Now Distribution VIII (*a*) indicated that, when marbles were used, the Distribution of Means of 1000 Samples of 10 obtained from (C) by actual sampling agreed well with the true Distribution of Means of Samples. Consequently the corresponding Distribution of σ^2 of these 1000 Samples was obtained.

VIII (*b*). The σ^2 of 1000 Samples of 10 from Population (C), the sampling being conducted by the use of coloured marbles.

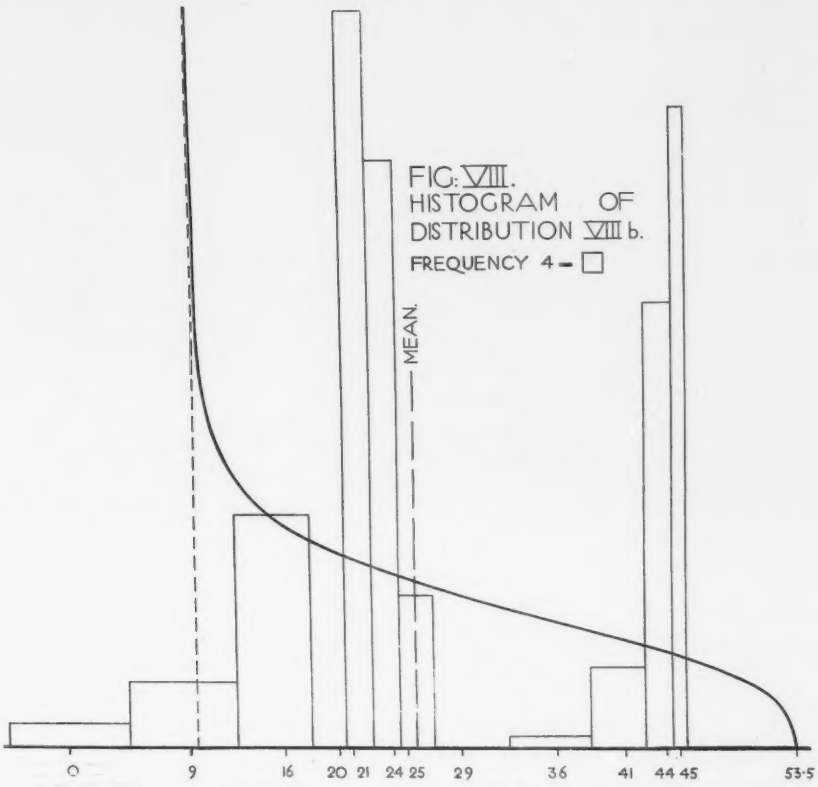
Value of σ^2	·00	·09	·16	·20	·21	·24	·25	·29	·36	·41	·44	·45
Frequency	15	76	188	0	214	171	56	0	10	47	130	93

The constants are:

Working origin = ·24. Working unit, the squared unit of (C).

$\nu_1' = {\cdot}018{,}820{,}0$, Mean = 0·258,820,0, $\Sigma_2 = 0{\cdot}120{,}080$,
$\nu_2' = {\cdot}014{,}773{,}4$, ${}_2M_2 = 0{\cdot}014{,}419{,}2$, ${}_2B_1 = 0{\cdot}194{,}083$,
$\nu_3' = {\cdot}001{,}583{,}6$, ${}_2M_3 = 0{\cdot}000{,}762{,}8$, ${}_2B_2 = 2{\cdot}108{,}627$,
$\nu_4' = {\cdot}000{,}526{,}3$, ${}_2M_4 = 0{\cdot}000{,}438{,}2$.

Note. In the representation of this distribution in Fig. VIII the unit of σ^2 is $\frac{1}{100}$ the squared unit of (C).



This is a discrete distribution like the true Distribution of σ^2 of Samples of 10 from (C) given before in this part, of which distribution it is itself a sample of course. If we reduce the total frequency of this true distribution of σ^2 from 254,186,856 to 1000, we can compare this sample with it. Carrying out this reduction, we have

Value of σ^2	Frequency in VIII (*b*)	True Frequency reduced by $\frac{1000}{254,186,856}$
·00	15	12·86
·09	76	80·37
·16	188	191·53
·20	0	0·00
·21	214	228·02
·24	171	156·76
·25	56	52·67
·29	0	0·14
·36	10	9·94
·41	47	55·09
·44	130	128·99
·45	93	83·61

Applying the Goodness of Fit test, we have

$$\chi^2 = 5{\cdot}41,\ P = {\cdot}9075 \text{ with } 12 \text{ Groups.}$$

This shows that VIII (*b*) is an excellent sample of the true Distribution of σ^2 of Samples of 10 from (C), and consequently an attempt has been made to compare VIII (*b*) with the *J*-curve.

Now VIII (*b*) is a discrete distribution with unequal intervals, so the frequency of any value of σ^2 was represented by a rectangle, whose base was the distance between the two points of bisection of the intervals on either side of this value of σ^2. The height of the rectangle being properly chosen the frequency of this value of σ^2 is thus represented by an area, which is the first step in the comparison because it is areas under the *J*-curve that represent frequencies. However, as the intervals between successive values of σ^2 in VIII (*b*) are unequal, this method causes the value of σ^2 considered *not* to be on the mid-ordinate of the rectangle whose area represents the frequency of the value. Thus while the representation is purely approximate and pictorial, it suffices to show that the *J*-curve in no way represents the Distribution of σ^2 of Samples of 10 from (C).

Summary of Conclusions reached.

(*a*) The Distribution of Means of Small Samples from an infinite Population.

(1) Even when the sampled population is quite skew, this distribution exhibits a surprising tendency to normality. Hence its approximate representation by a normal curve will be quite good in general and the assumption that, for small samples as well as large, the distribution of means from a quite skew infinite population is normal will be quite useful in practical statistics.

(2) By the aid of moment-coefficients a good representation of this distribution by a Pearson Curve can be obtained showing quite well the skewness, small as this is in general.

(3) The true curve of means of samples is known when the sampling is from a population represented by a Pearson Type III Curve.

(*b*) The Distribution of σ^2 of Small Samples from an infinite Population.

(1) Tchouproff's corrected formulae for the moment-coefficients of this distribution will lead, with the aid of a Pearson Curve, to a good representation of the distribution in practical cases.

(2) The labour involved in the calculation of the higher moment-coefficients of the sampled population, necessitated by the exact use of Tchouproff's formulae, can be obviated by the assumption that a Pearson Curve will represent the population, which enables approximate values for these higher moment-coefficients to be calculated rapidly by the well-known difference formulae. This approximation does not appear to affect the accuracy of the representation of this distribution of σ^2 of samples to any appreciable extent.

(*c*) The Distribution of Means of Small Samples from a finite Population.

(1) Here again, even when the sampled population is very skew and as small as 36, the Distribution of Means of Samples still shows a very strong tendency to normality. In fact, it appears that the assumption of the normality of this distribution is legitimate for practical purposes, no matter what the size of the sample or the sampled population, and no matter upon which basis the sampling is conducted. It is only likely to fail as a practical hypothesis if the sampled population follows a really extreme form of skewness.

(2) This distribution can be represented satisfactorily by a Pearson Curve based on the formulae for its moment-coefficients given in Part IV.

(3) Subject to conditions that have been stated broadly, it has been possible to show that this distribution can be represented approximately by a Pearson Curve based upon the values of its moment-coefficients which result from the assumption that the sampled population is infinite in all its categories. That is to say, it has been possible to indicate the conditions under which, in dealing with the Distribution of Means of Samples, it is immaterial whether the sampling be conducted by drawing each time complete samples or individuals only before replacements.

(*d*) The Distribution of σ^2 of Small Samples from a finite Population.

(1) The two formulae for the 3rd and 4th moment-coefficients of this distribution have been obtained, and thus with Dr Neyman's results the first four moments of the distribution are known. The great length of these two formulae however contributes materially to the difficulty of accurate representation of this distribution.

(2) When the sampled population is large or of moderate size, the length of the formulae for ${}_2M_3$ and ${}_2M_4$, together with their general unwieldiness, makes the evaluation of these constants by their aid almost an arithmetical impossibility at any rate in practical statistics. We are thus compelled to resort to an approximate representation of the distribution by assuming the sampled population infinite and calculating approximate values for the moment-coefficients of the distribution by means of Tchouproff's formulae as in Part III. This approximation is probably quite good when the total frequency of the sampled population is of the order of several thousands, but for moderate-sized sampled populations with total frequency several hundreds its usefulness and accuracy are certainly questionable.

(3) When the sampled population is very small in its total frequency, the formulae for the moment-coefficients of the Distribution of σ^2 of Samples give the values of these constants quite accurately and with moderate ease, but the representation by a Pearson Curve is liable to fail. Since the sampled population when it is small, has only a few categories, the distribution of σ^2 may become multi-modal and hence, as the Pearson Curves involve the assumption of a uni-modal frequency, they are naturally incapable of representing the distribution; whilst further, not only does it tend to become one with discrete frequencies, but these

frequencies occur at unequal intervals, rendering its representation by any curve a matter of doubtful validity and accuracy.

(*e*) Considerations upon the actual practice of sampling.

When the actual sampling is conducted on the basis of an infinite sampled population, that is, individuals only are drawn before replacement, the use of one sample whose randomness is established, such as Mr L. C. Tippett's Random Numbers, provides a really reliable method. Further, it is one which takes precedence over any system of drawing objects from a bag, bowl or urn because of its much greater speed and ease in operation.

When the sampling is conducted on the basis of a finite sampled population, that is, complete samples are drawn each time before replacement, or when a typical random sample is not available, the use of marked tickets appears to be of little or no value because of the almost insuperable difficulty of obtaining really effective shuffling between consecutive draws. If the sampled population be very large, roughly homogeneous and symmetrical material such as commercial coloured glass beads, which provide easy means of distinguishing between the various categories of the sampled population, appears to be quite satisfactory in practice. If however the sampled population be quite small, the lack of true homogeneity even in such material as this will cause bias in actual sampling and, if good results are to be obtained, some material the individuals of which really approximate quite closely to the ideal of complete homogeneity must be employed.

TABLE OF THE RATIO: AREA TO BOUNDING ORDINATE, FOR ANY PORTION OF NORMAL CURVE.

By JOHN P. MILLS.

*Introduction**. It has been recognized for some time that a table of the ratio, area to ordinate, for the normal curve would be useful. It was pointed out, for example, in *Biometrika*, Vol. XVI. p. 164, that the area A of the tail of a frequency curve could be found approximately by the formula $A = \phi(0)\mathcal{R}_x$, where $\phi(0)$ is the ordinate of the curve at the stump and x and $\mathcal{R}_x$ depend on the first and second derivatives at that point. Particular examples of application to the point binomial were given in succeeding pages of that volume, and to the hypergeometric series in Vol. XVII. pp. 61 *et seq.* In all these cases the need for a tabulation of $\mathcal{R}_x$ was apparent because for certain values of x its value could not be obtained easily and with sufficient accuracy from existing tables of the area and the ordinate. The ratio $\mathcal{R}_x$ is also important because its reciprocal equals the mean value of the area of the tail, from x onwards, of the normal curve $y = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$. The methods by which this table were computed were different in different parts. For the interval, $0 \leqq x \leqq 1{\cdot}84$, Sheppard's well-known tables gave correct results, save in six cases, which were examined separately by the methods used later. For $1{\cdot}84 \leqq x \leqq 4{\cdot}20$ it was necessary to use the tables of Dr Jas. Burgess, *Trans. R. S. Edinburgh*, Vol. XXXIX. He tabulated

$$H_t = \frac{2}{\sqrt{\pi}} \int_0^t e^{-t^2}\, dt, \quad \log \zeta_t = \frac{2}{\sqrt{\pi}} e^{-t^2}.$$

If
$$\rho_t = \frac{1 - H_t}{\zeta_t}, \text{ then } \mathcal{R}_x = \rho_t \sqrt{2} \text{ at } t = x/\sqrt{2}.$$

Accordingly ρ was first computed, the values of ζ being taken from an eight place logarithm table, and the divisions made on a machine. For $1{\cdot}300 \leqq t \leqq 1{\cdot}500$, H and ζ were extracted at intervals of $\Delta t = 0{\cdot}005$, and ρ_t and $\mathcal{R}_x$ computed. This gave a provisional table of $\mathcal{R}_x$ at the inconvenient points, $1{\cdot}300\sqrt{2}$, $1{\cdot}305\sqrt{2}$, etc. What was desired was a set of values at $x = 1{\cdot}80$, $1{\cdot}81$, etc. These could be found by interpolation from the provisional table already obtained, but to produce the desired accuracy easily, it was desirable first to obtain twice as many values of $\mathcal{R}_x$ at intervals half as great. These were found quickly by Pearson's 4-point mid-panel formula. For the interval, $1{\cdot}50 \leqq t \leqq 2{\cdot}15$, sufficient accuracy could be obtained by first extracting t at intervals of $0{\cdot}01$, and then halving them as before. For $2{\cdot}15 \leqq t \leqq 3{\cdot}00$, a somewhat greater simplification was possible. By computing several values of ρ_t near the point $t = 2{\cdot}20\sqrt{2}$ the value of $\mathcal{R}_x$ was found by interpolation at $2{\cdot}20$; and then, in similar manner, at $2{\cdot}30$, $2{\cdot}40$, etc. Then the intervening tenths of the x interval were filled in by using Pearson's 5-point formula. From $x = 4{\cdot}25$ ($t = 3$) to the end, the values were supplied by the Editor

* By B. H. Camp, under whose direction this table was prepared.

from material already available at the Galton Laboratory. Some of this part was also computed by the author from the formula

$$\mathcal{R}_x = \frac{1}{x}\left(1 - \frac{1}{x^2} + \frac{1.3}{x^4} - \frac{1.3.5}{x^6} + \ldots\right).$$

The entire computation was checked by differencing, and by overlapping the several methods. The errors introduced were investigated. Eight places of decimals were retained throughout, and the original computation therefore gave a table of $\mathcal{R}_x$ to eight places. In no case did the possible error in the eighth place jeopardize the accuracy of the fifth place retained in the final table; save only in the six instances already mentioned near the beginning of the table, and these were eventually calculated by the later methods to nine places.

$$\left(\textit{The Ratio } \mathcal{R}_x = e^{\frac{1}{2}x^2}\int_x^\infty e^{-\frac{1}{2}x^2}dx.\right)$$

x	$\mathcal{R}_x$	$-\Delta$	x	$\mathcal{R}_x$	$-\Delta$
·00	1·25331	993	·40	·93567	623
·01	1·24338	982	·41	·92944	615
·02	1·23356	969	·42	·92329	609
·03	1·22387	957	·43	·91720	602
·04	1·21430	946	·44	·91118	596
·05	1·20484	934	·45	·90522	590
·06	1·19550	923	·46	·89932	583
·07	1·18627	911	·47	·89349	577
·08	1·17716	900	·48	·88772	571
·09	1·16816	890	·49	·88201	565
·10	1·15926	878	·50	·87636	558
·11	1·15048	869	·51	·87078	553
·12	1·14179	858	·52	·86525	548
·13	1·13321	847	·53	·85977	541
·14	1·12474	838	·54	·85436	536
·15	1·11636	827	·55	·84900	530
·16	1·10809	818	·56	·84370	525
·17	1·09991	808	·57	·83845	519
·18	1·09183	799	·58	·83326	514
·19	1·08384	790	·59	·82812	509
·20	1·07594	780	·60	·82303	504
·21	1·06814	771	·61	·81799	498
·22	1·06043	762	·62	·81301	494
·23	1·05281	754	·63	·80807	488
·24	1·04527	745	·64	·80319	484
·25	1·03782	736	·65	·79835	478
·26	1·03046	728	·66	·79357	474
·27	1·02318	719	·67	·78883	469
·28	1·01599	712	·68	·78414	465
·29	1·00887	703	·69	·77949	460
·30	1·00184	696	·70	·77489	455
·31	0·99488	687	·71	·77034	451
·32	·98801	680	·72	·76583	446
·33	·98121	673	·73	·76137	442
·34	·97448	665	·74	·75695	438
·35	·96783	657	·75	·75257	433
·36	·96126	651	·76	·74824	430
·37	·95475	643	·77	·74394	425
·38	·94832	636	·78	·73969	421
·39	·94196	629	·79	·73548	417

TABLE (*continued*).

x	$\mathcal{R}_x$	$-\Delta$	x	$\mathcal{R}_x$	$-\Delta$
·80	·73131	413	1·30	·56487	265
·81	·72718	409	1·31	·56222	262
·82	·72309	405	1·32	·55960	261
·83	·71904	401	1·33	·55699	258
·84	·71503	397	1·34	·55441	256
·85	·71106	394	1·35	·55185	254
·86	·70712	390	1·36	·54931	252
·87	·70322	387	1·37	·54679	249
·88	69935	382	1·38	·54430	248
·89	·69553	380	1·39	·54182	246
·90	·69173	375	1·40	·53936	244
·91	·68798	373	1·41	·53692	242
·92	·68425	368	1·42	·53450	240
·93	·68057	366	1·43	·53210	238
·94	·67691	362	1·44	·52972	237
·95	·67329	358	1·45	·52735	234
·96	·66971	356	1·46	·52501	233
·97	66615	352	1·47	·52268	230
·98	·66263	349	1·48	·52038	229
·99	·65914	346	1·49	·51809	227
1·00	·65568	343	1·50	·51582	226
1·01	·65225	340	1·51	·51356	223
1·02	·64885	336	1·52	·51133	222
1·03	·64549	334	1·53	·50911	221
1·04	·64215	330	1·54	·50690	218
1·05	·63885	328	1·55	·50472	217
1·06	·63557	325	1·56	·50255	215
1·07	·63232	322	1·57	·50040	214
1·08	·62910	319	1·58	·49826	212
1·09	·62591	317	1·59	·49614	210
1·10	·62274	313	1·60	·49404	209
1·11	·61961	311	1·61	·49195	207
1·12	·61650	308	1·62	·48988	206
1·13	·61342	306	1·63	·48782	204
1·14	·61036	303	1·64	·48578	202
1·15	·60733	300	1·65	·48376	201
1·16	·60433	298	1·66	·48175	200
1·17	·60135	295	1·67	·47975	198
1·18	·59840	292	1·68	·47777	197
1 19	·59548	291	1·69	·47580	195
1·20	·59257	287	1·70	·47385	193
1·21	·58970	286	1·71	·47192	193
1·22	·58684	282	1·72	·46999	191
1·23	·58402	281	1·73	·46808	189
1·24	·58121	278	1·74	·46619	188
1·25	·57843	276	1·75	·46431	187
1·26	·57567	273	1·76	·46244	186
1·27	·57294	272	1·77	·46058	184
1·28	·57022	268	1·78	·45874	182
1·29	·56754	267	1·79	·45692	182

TABLE (*continued*).

x	$\mathcal{R}_x$	$-\Delta$	x	$\mathcal{R}_x$	$-\Delta$
1·80	·45510	180	2·30	·37858	129
1·81	·45330	179	2·31	·37729	128
1·82	·45151	178	2·32	·37601	127
1·83	·44973	176	2·33	·37474	126
1·84	·44797	175	2·34	·37348	126
1·85	·44622	174	2·35	·37222	125
1·86	·44448	173	2·36	·37097	124
1·87	·44275	171	2·37	·36973	123
1·88	·44104	170	2·38	·36850	123
1·89	·43934	169	2·39	·36727	122
1·90	·43765	168	2·40	·36605	121
1·91	·43597	167	2·41	·36484	120
1·92	·43430	165	2·42	·36364	120
1·93	·43265	165	2·43	·36244	119
1·94	·43100	163	2·44	·36125	118
1·95	42937	162	2·45	·36007	118
1·96	·42775	161	2·46	·35889	116
1·97	·42614	160	2·47	·35773	116
1·98	·42454	159	2·48	·35657	116
1·99	·42295	158	2·49	·35541	114
2·00	·42137	157	2·50	·35427	114
2·01	·41980	155	2·51	·35313	114
2·02	·41825	155	2·52	·35199	112
2·03	·41670	154	2·53	·35087	112
2·04	·41516	152	2·54	·34975	112
2·05	·41364	152	2·55	·34863	110
2·06	·41212	150	2·56	·34753	110
2·07	·41062	150	2·57	·34643	110
2·08	·40912	148	2·58	·34533	108
2·09	·40764	148	2·59	·34425	109
2·10	·40616	146	2·60	·34316	107
2·11	·40470	146	2·61	·34209	107
2·12	·40324	145	2·62	·34102	106
2·13	·40179	143	2·63	·33996	106
2·14	·40036	143	2·64	·33890	105
2·15	·39893	142	2·65	·33785	104
2·16	·39751	141	2·66	·33681	104
2·17	·39610	140	2·67	·33577	103
2·18	·39470	139	2·68	·33474	103
2·19	·39331	138	2·69	·33371	102
2·20	·39193	138	2·70	·33269	101
2·21	·39055	136	2·71	·33168	101
2·22	·38919	136	2·72	·33067	100
2·23	·38783	134	2·73	·32967	100
2·24	·38649	134	2·74	·32867	98
2·25	·38515	133	2·75	·32768	98
2·26	·38382	132	2·76	·32669	98
2·27	·38250	132	2·77	·32571	97
2·28	·38118	130	2·78	·32474	97
2·29	·37988	130	2·79	·32377	97

TABLE (*continued*).

x	$\mathcal{R}_x$	$-\Delta$	x	$\mathcal{R}_x$	$-\Delta$
2·80	·32280	96	3·30	·28064	74
2·81	·32184	95	3·31	·27990	73
2·82	·32089	95	3·32	·27917	73
2·83	·31994	94	3·33	·27844	72
2·84	·31900	94	3·34	·27772	73
2·85	·31806	93	3·35	·27699	72
2·86	·31713	93	3·36	·27627	71
2·87	·31620	92	3·37	·27556	71
2·88	·31528	92	3·38	·27485	71
2·89	·31436	91	3·39	·27414	71
2·90	·31345	91	3·40	·27343	70
2·91	·31254	90	3·41	·27273	70
2·92	·31164	90	3·42	·27203	69
2·93	·31074	89	3·43	·27134	69
2·94	·30985	89	3·44	·27065	69
2·95	·30896	88	3·45	·26996	69
2·96	·30808	88	3·46	·26927	68
2·97	·30720	88	3·47	·26859	68
2·98	·30632	87	3·48	·26791	67
2·99	·30545	86	3·49	·26724	67
3·00	·30459	86	3·50	·26657	67
3·01	·30373	86	3·51	·26590	67
3·02	·30287	85	3·52	·26523	66
3·03	·30202	84	3·53	·26457	66
3·04	·30118	84	3·54	·26391	65
3·05	·30034	84	3·55	·26326	66
3·06	·29950	84	3·56	·26260	65
3·07	·29866	82	3·57	·26195	64
3·08	·29784	83	3·58	·26131	65
3·09	·29701	82	3·59	·26066	64
3·10	·29619	81	3·60	·26002	63
3·11	·29538	82	3·61	·25939	64
3·12	·29456	80	3·62	·25875	63
3·13	·29376	81	3·63	·25812	63
3·14	·29295	80	3·64	·25749	63
3·15	·29215	79	3·65	·25686	62
3·16	·29136	79	3·66	·25624	62
3·17	·29057	79	3·67	·25562	62
3·18	·28978	78	3·68	·25500	61
3·19	·28900	78	3·69	·25439	61
3·20	·28822	78	3·70	·25378	61
3·21	·28744	77	3·71	·25317	61
3·22	·28667	77	3·72	·25256	60
3·23	·28590	76	3·73	·25196	60
3·24	·28514	76	3·74	·25136	60
3·25	·28438	75	3·75	·25076	59
3·26	·28363	76	3·76	·25017	60
3·27	·28287	74	3·77	·24957	59
3·28	·28213	75	3·78	·24898	58
3·29	·28138	74	3·79	·24840	59

TABLE (*continued*).

x	$\mathcal{R}_x$	$-\Delta$	x	$\mathcal{R}_x$	$-\Delta$
3·80	·24781	58	6·00	·16238	253
3·81	·24723	58	6·10	·15984	246
3·82	·24665	58	6·20	·15739	239
3·83	·24607	57	6·30	·15500	232
3·84	·24550	57	6·40	·15269	225
3·85	·24493	57	6·50	·15044	218
3·86	·24436	57	6·60	·14825	212
3·87	·24379	56	6·70	·14613	206
3·88	·24323	56	6·80	·14407	201
3·89	·24267	56	6·90	·14206	195
3·90	·24211	56	7·00	·14010	190
3·91	·24155	55	7·10	·13820	185
3·92	·24100	55	7·20	·13635	180
3·93	·24045	55	7·30	·13455	176
3·94	·23990	55	7·40	·13279	171
3·95	·23935	54	7·50	·13108	167
3·96	·23881	55	7·60	·12941	163
3·97	·23826	54	7·70	·12778	159
3·98	·23772	53	7·80	·12619	155
3·99	·23719	54	7·90	·12464	151
4·00*	·23665	264	8·00	·12313	148
4·05	·23401	258	8·10	·12166	144
4·10	·23143	253	8·20	·12021	141
4·15	·22890	248	8·30	·11880	138
4·20	·22642	243	8·40	·11743	135
4·25	·22399	238	8·50	·11608	132
4·30	·22161	233	8·60	·11477	129
4·35	·21928	228	8·70	·11348	126
4·40	·21700	224	8·80	·11222	123
4·45	·21476	219	8·90	·11099	120
4·50	·21257	215	9·00	·10979	118
4·55	·21042	211	9·10	·10861	115
4·60	·20831	207	9·20	·10745	113
4·65	·20624	203	9·30	·10632	111
4·70	·20421	199	9·40	·10522	108
4·75	·20222	195	9·50	·10413	106
4·80	·20027	192	9·60	·10307	104
4·85	·19835	188	9·70	·10203	102
4·90	·19647	185	9·80	·10101	100
4·95	·19462	181	9·90	·10001	98
5·00*	·19281	353	10·00	·09903	
5·10	·18928	341			
5·20	·18587	329			
5·30	·18258	318			
5·40	·17940	307			
5·50	·17632	297			
5·60	·17335	288			
5·70	·17047	278			
5·80	·16769	270			
5·90	·16499	261			

* Note the change in the x interval.

ON THE CORRELATION OF THE MEAN AND THE VARIANCE IN SAMPLES DRAWN FROM AN "INFINITE" POPULATION

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1. The relation between the two characters in a sample, mean $\bar{x}$ and standard deviation σ (or variance σ^2), is a very interesting one as several methods of judging whether a certain sample has been taken from a given population, or if two different samples have been taken from the same population, are based on the comparison of certain functions of $\bar{x}$ with certain functions of σ.

For the case where the original frequency distribution is normal, the question has been largely discussed, in particular in the paper of "Student"* in which the probability is found of having different values of $\frac{\bar{x}}{\sigma}$, where $\bar{x}$ is measured from the mean of the sampled population. We know that in the case of a normal population $\bar{x}$ and σ are independent, but we know further that the populations having a limited range which can be met with in practice are, speaking rigorously, never normal. In many cases, also, we are dealing with populations which differ from the normal ones in other characters besides the range. It is interesting therefore to consider the relation of $\bar{x}$ and σ for such populations in order to get some idea about the possible differences in these properties.

In dealing with this matter we may use the results of an earlier paper by the present author about skew regression published in this volume of Biometrika, and also the formula of K. Pearson giving the second order parabola of regression†. After deducing a second order parabola of regression of σ^2 on $\bar{x}$ in a sample taken from an indefinitely large population with any given distribution, we shall show that the calculation of higher parabolae is of little value, as their expressions for small values of n—the number of individuals in a sample—are too complicated for us to draw practical conclusions from. Further the difference between them and the expression of the second order parabola vanishes as $n \to \infty$.

Afterwards we shall draw attention to a method of judging whether a sample is likely to have been taken from a population whose distribution is supposed to be known.

2. We shall use a notation like that in the previous paper, namely: μ_k will denote the kth moment of the sampled population about its mean; λ_k—the mean value of $(\sigma^2 - \overline{\sigma}^2)\bar{x}^k$ and n—the number of individuals in the sample.

* *Biometrika*, Vol. VI. pp. 1 *et seq.*

† *Biometrika*, Vol. XIII. pp. 296—300.

It is convenient to measure the deviations of the two variates σ^2 and $\bar{x}$ from their respective means in terms of their standard deviations, namely

$$\sigma_{\bar{x}} = \sqrt{\frac{\mu_2}{n}}, \quad \sigma_{\sigma^2} = \frac{\mu_2}{n}\sqrt{\frac{n-1}{n}[(n-1)\beta_2 - n + 3]} \quad \text{.....(1).}$$

Adopting these units we shall put in our formulae for the higher parabolae of regression, instead of λ_k, the expression

$$q_k = \frac{\lambda_k}{\sigma_{\sigma^2}\,\sigma_{\bar{x}}{}^k} \quad \text{.....(2).}$$

The deviation of the mean in any sample from its mean value in samples we shall denote by $\bar{x}'$ and the ordinates of the parabola of regression measured from the mean of σ^2 by u'; when $\bar{x}'$ and u' are measured in terms of their standard deviations as given in (1) we shall term them $\bar{x}$ and u. We shall also need the usual frequency constants β for the sampled population and for the population of means of samples taken from the former. The first we shall denote by $\beta_1, \beta_2, \ldots$ and the second by $B_1, B_2, \ldots$.

The formula for the second order parabola of regression can be taken directly from the paper of K. Pearson (*loc. cit.*), and applying it to the variates under consideration we have

$$u = q_1\bar{x} + \frac{q_2 - q_1\sqrt{B_1}}{B_2 - B_1 - 1}(\bar{x}^2 - \sqrt{B_1}\,\bar{x} - 1) \quad \text{.....(3).}$$

It remains to express all the constants q_k and the B's in terms of the β's. All necessary formulae are contained in my paper in *Biometrika*, Vol. XVII. pp. 472—479. We have

$$H_1 = q_1 = \rho' = \frac{\sqrt{n-1}\ \sqrt{\beta_1}}{\sqrt{(n-1)\beta_2 - n + 3}} \quad \text{.....(4),}$$

$$q_2 = \frac{R'\sigma_1}{\sigma_{\bar{x}}{}^2} = \sqrt{\frac{n-1}{n}}\,\frac{\beta_2 - 3}{\sqrt{(n-1)\beta_2 - n + 3}} \quad \text{.....(5),}$$

where ρ', R' and σ are symbols previously used.

Further $$B_1 = \frac{\beta_1}{n}, \quad B_2 = \frac{3(n-1) + \beta_2}{n} \quad \text{.....(6).}$$

Using these formulae, we have

$$u = \frac{\sqrt{n-1}\ \sqrt{\beta_1}}{\sqrt{(n-1)\beta_2 - n + 3}}\bar{x} + \frac{\sqrt{n(n-1)}}{2(n-1) + \beta_2 - \beta_1 - 1}\,\frac{\beta_2 - \beta_1 - 3}{\sqrt{(n-1)\beta_2 - n + 3}}\left(\bar{x}^2 - \sqrt{\frac{\beta_1}{n}}\,\bar{x} - 1\right)$$

.....(7),

and the square of the corresponding second order correlation ratio is

$$H_2{}^2 = \frac{(n-1)\beta_1}{(n-1)\beta_2 - n + 3} + \frac{n-1}{2(n-1) + \beta_2 - \beta_1 - 1}\,\frac{(\beta_2 - \beta_1 - 3)^2}{(n-1)\beta_2 - n + 3}$$

$$= \frac{(n-1)\beta_1}{(n-1)\beta_2 - n + 3}\left\{1 + \frac{(\beta_2 - \beta_1 - 3)^2}{\beta_1(2n + \beta_2 - \beta_1 - 3)}\right\} \quad \text{.....(8a),}$$

$$= 1 - \frac{2n^2(\beta_2 - \beta_1 - 1)}{\{(n-1)\beta_2 - n + 3\}\{2n + \beta_2 - \beta_1 - 3\}} \quad \text{.....(8b).}$$

From (8a) we see that for large values of n the second order term has no importance provided β_1 is not zero. If $\beta_1 = 0$ the first order term disappears and the correlation depends only on the remaining term, unless $\beta_2 = 3$, at the Normal Point, when this term also vanishes. When $n \to \infty$, then $H_2^2 \to \frac{\beta_1}{\beta_2 - 1}$, that is to say the contours in the β_1, β_2 field of equal correlation between $\bar{x}$ and σ^2 are approximately straight lines radiating from the point $\beta_2 = 1$, $\beta_1 = 0$. Along the boundary line $\beta_1 = 0$ we have seen that $H_1^2 = 0$, while the form (8b) shows that along the other boundary of the field, namely, $\beta_2 - \beta_1 - 1 = 0$, there is perfect relationship, or $H_2^2 = 1$.

This latter equality arises from the well-known fact that the populations corresponding to β_1, β_2 points along this line contain only two classes of individuals; the same condition must hold in the sample, where let us say there are p individuals with character value a and $n - p$ with character value b. Then the mean and the variance of the sample will be definite functions of p, namely

$$\bar{x} = \frac{pa + (n-p)b}{n}, \quad \sigma^2 = \frac{(n-p)p}{n^2}(a+b)^2 \ldots\ldots\ldots\ldots\ldots\ldots (9),$$

whence

$$\sigma^2 = \left(\frac{a+b}{b-a}\right)^2 (b - \bar{x})(\bar{x} - a) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10).$$

If now we express a and b in terms of β_1 and $\beta_2 = \beta_1 + 1$, we shall be able to transform (10) into the regression parabola (7) corresponding to these values of the β's.

Turning to (7), it will be noticed that the sign of the coefficient of $\bar{x}^2$ depends upon the sign of $\kappa = \beta_2 - \beta_1 - 3$. That is to say for samples from populations whose distributions correspond to β's lying above* the line $\beta_2 - \beta_1 - 3 = 0$, the two infinite branches of the regression parabola are directed in the negative sense of the u-axis, while for samples from populations with β's lying on this line the second order term vanishes and (7) becomes linear; and for samples from populations with β's below this line, the branches of the parabola are directed in the positive sense. $\beta_2 - \beta_1 - 3 = 0$ is the line along which lie the β's of distributions representing Poisson's limit to the binomial, and it also divides the areas of positive and negative binomials.

Let us now consider the form of (7) a little more in detail. In general (that is to say, when the original frequency distribution is not a U- or J-shaped one) the most interesting points of the x-axis are those near the mean and the mode, and we shall consider the behaviour of the second order parabola of regression near these points. The abscissa of the mode of means in samples is not known, but we can take, as an approximation to it, the abscissa $M(\bar{x})$ of the mode of a Pearson curve fitted to the distribution of means. It is given by

$$M(\bar{x}) = -\frac{\sqrt{B_1}(B_2+3)}{2(5B_2 - 6B_1 - 9)} = -\sqrt{\frac{\beta_1}{n}}\,\frac{6(n-1)+\beta_2+3}{2[6(n-1)+5\beta_2 - 6\beta_1 - 9]} \ldots\ldots (11).$$

* Above, in the sense of Fig. 2.

The sign of $M(\bar{x})$ depends upon the sign of $\sqrt{\beta_1}\,[6(n-1)+5\beta_2-6\beta_1-9]$. The expression in brackets is generally positive. In fact, if it be negative, then $5\beta_2-6\beta_1-9$ must be negative, which happens when the original frequency distribution is a J-shaped or a U-shaped one and is unusual. Further, as $\beta_2 \geqslant \beta_1+1$,

$$6(n-1)+5\beta_2-6\beta_1-9 \geqslant 6n-\beta_1-10$$

and if the left-hand side be negative then the right-hand one must be so and therefore $\beta_1 \geqslant 6n-10$, which for any value of $n \geqslant 2$ gives values of β_1 very unlikely to be met with in practice. Therefore we shall consider only the case, when the denominator in (11) is positive and so that the sign of $M(\bar{x})$ is that of $-\sqrt{\beta_1}$.

Let ξ be the abscissa of the vertex of the parabola (7). We find easily:

$$\xi = -\frac{\sqrt{n\beta_1}}{\beta_2-\beta_1-3}.$$

If $\beta_2-\beta_1-3<0$ then the signs of $M(\bar{x})$ and ξ are different and thus the mean of the population is between the mode $M(\bar{x})$ and the vertex of the parabola. We remember that in this case the infinite branches of the parabola are directed in the negative sense of the u-axis.

Suppose now that $\beta_2-\beta_1-3>0$. Then the signs of ξ and $M(\bar{x})$ are the same. Let us consider

$$\frac{\xi}{M(\bar{x})} = \frac{2n[6(n-1)+5\beta_2-6\beta_1-9]}{(\beta_2-\beta_1-3)[6(n-1)+\beta_2+3]} \quad \ldots\ldots\ldots\ldots\ldots\ldots(12),$$

and find the condition that it be less than unity.

This condition can be written

$$12n^2+2n(5\beta_2-6\beta_1-15)<6n(\beta_2-\beta_1-3)+(\beta_2-3)(\beta_2-\beta_1-3),$$

and we see that it is satisfied if n lies between the two roots of the equation

$$12n^2+2n(2\beta_2-3\beta_1-6)-(\beta_2-3)(\beta_2-\beta_1-3)=0,$$

that is, (as n is positive), when

$$n<\frac{\sqrt{(2\beta_2-3\beta_1-6)^2+12(\beta_2-3)(\beta_2-\beta_1-3)}-(2\beta_2-3\beta_1-6)}{12}.$$

As $0 \leqslant \beta_1 \leqslant \beta_2-3$, it follows that for $\xi < M(\bar{x})$ it is necessary that

$$n<\frac{5(\beta_2-3)}{12} \quad \text{or} \quad \beta_2>\frac{12n+15}{5} \geqslant 7{\cdot}8.$$

In practice we meet very rarely a β_2 with so high a value as 7·8 (which corresponds to the case $n=2$!) and therefore see that with rare exceptions the mode $M(\bar{x})$ lies between the vertex and the mean of the population. Further, in the case we are considering ($\beta_2-\beta_1-3>0$) the infinite branches of the parabola are directed in the positive sense of the u-axis.

We conclude therefore that for samples from populations likely to be met with in practice, that is to say when

$$n>\frac{\beta_1+10}{6}, \quad \text{and} \quad n>\frac{5(\beta_2-3)}{12},$$

the ordinate, u, of the regression parabola of the variance decreases as we pass from the mean value of $\bar{x}$ to its modal value in samples. Further if the size of the sample n is considerable, since the ratio $\xi/M(\bar{x})$ is of order n, the variation of u in the neighbourhood of the modal and mean $\bar{x}$ will be approximately linear. The position can be illustrated diagrammatically as in Fig. 1, which corresponds to the case of a sample drawn from a population with positive skewness (μ_3 or $\sqrt{\beta_1}$ positive) and having $\beta_2 - \beta_1 - 3 < 0$.

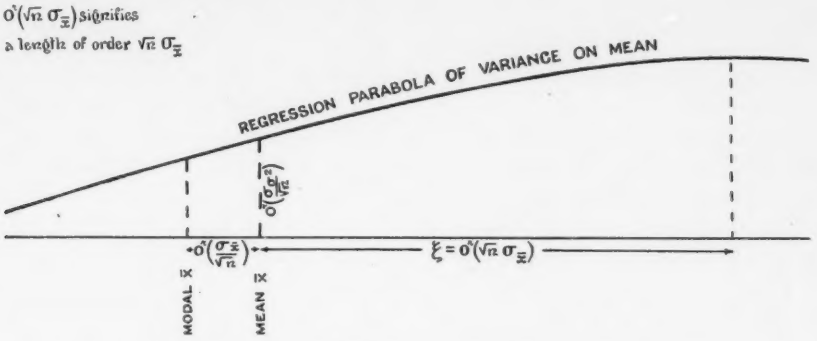

Fig. 1. Regression of Variance on Mean in Sample of n.

It is in fact only in small samples that the regression curve diverges far from a straight line within the range of significant frequency, i.e. within $\bar{x} = \pm 3\sigma_{\bar{x}}$, and $u = \pm 3\sigma_{\sigma^2}$, say. This is brought out clearly in the Tables on pp. 411—413, which give the values of $H_1 = |r|$, H_2, $100 \times \dfrac{H_2^2 - H_1^2}{H_2^2}$ for samples of 10, 50, 100 and 1000, and for population values of β_1 and β_2 covering the range commonly met with. Table III shows how the second order term becomes of slight importance for samples over 100, except for very small values of β_1 and for extreme values of β_2. Tables I and II make it clear that we may be very far wrong when considering a sample from a population of unknown form, if we assume that the normal rule of no correlation between the mean and the standard deviation in the sample is applicable.

Let us consider the locus of points on the $\beta_1\beta_2$ field corresponding to a given value of H_2. We have from (8b)

$$2n^2(\beta_2 - \beta_1 - 1) = (1 - H_2^2)[(n-1)\beta_2 - n + 3][2n + \beta_2 - \beta_1 - 3] \ldots (13),$$

which is the equation of a hyperbola passing through the point

$$\beta_1 = -\frac{2}{n-1}, \quad \beta_2 = \frac{n-3}{n-1}.$$

This hyperbola is independent of H_2, and has the asymptotes

$$(1 - H_2^2)[(n-1)\beta_2 - n + 3] = 2n^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (14),$$

$$\beta_2 - \beta_1 + 2n - 3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (15).$$

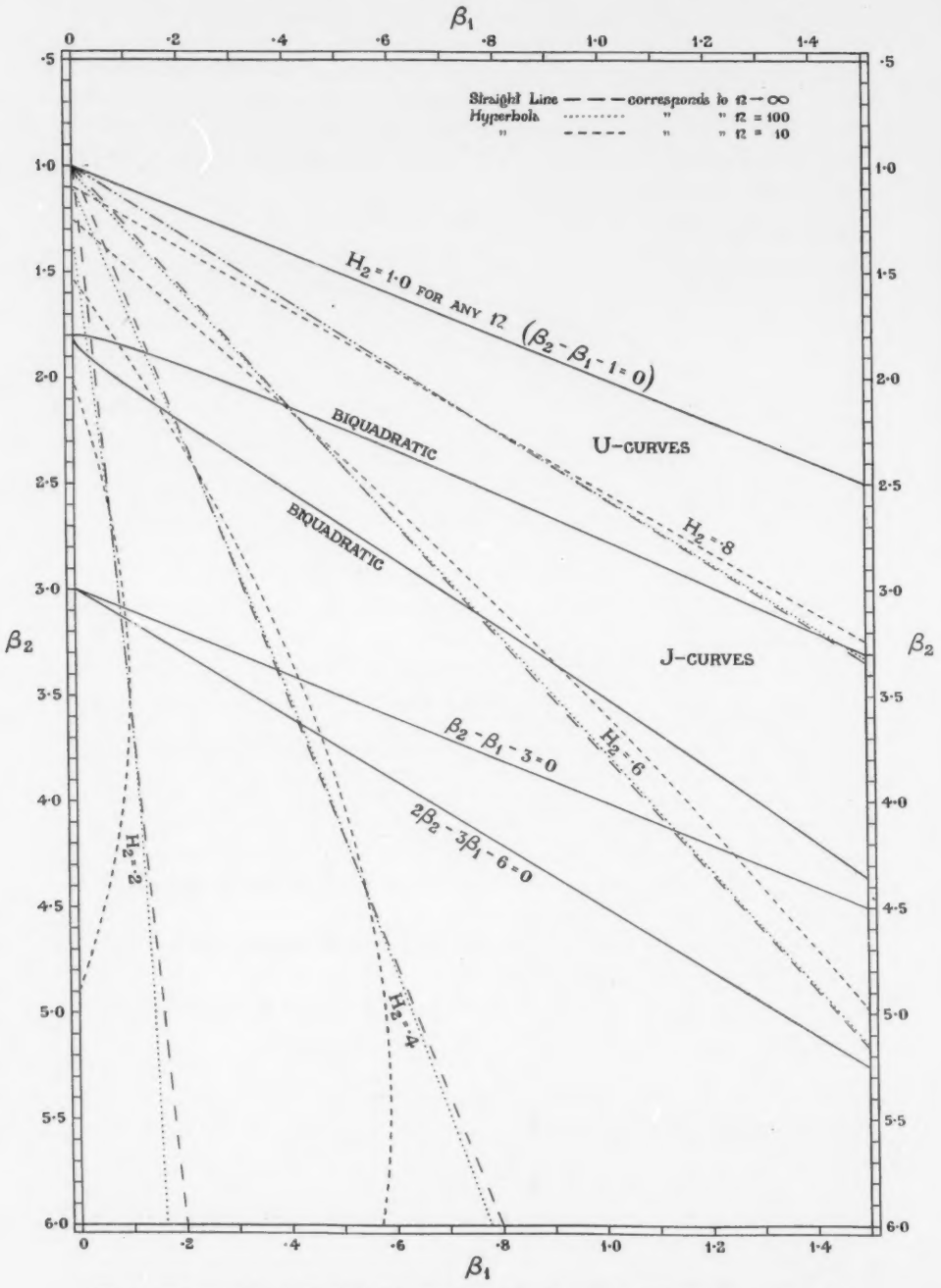
β_1
0 ·2 ·4 ·6 ·8 1·0 1·2 1·4
Straight Line — — — corresponds to n = ∞
Hyperbola ····· " " n = 100
" - - - - - " " n = 10
$H_2 = 1{\cdot}0$ FOR ANY n $(\beta_2 - \beta_1 - 1 = 0)$
BIQUADRATIC
U-CURVES
BIQUADRATIC
J-CURVES
$H_2 = 8$
$H_2 = 6$
$H_2 = 4$
$H_2 = 2$
$\beta_2 - \beta_1 - 3 = 0$
$2\beta_2 - 3\beta_1 - 6 = 0$
β_2
·5 1·0 1·5 2·0 2·5 3·0 3·5 4·0 4·5 5·0 5·5 6·0

Fig. 2. Diagram of β_1, β_2 Field showing Contours of Constant H_2 for Different Sizes of Samples.

If $n \to \infty$ then this hyperbola degenerates into a straight line

$$\beta_1 = H_2^2(\beta_2 - 1) \quad \text{.....(16),}$$

passing through the point $\beta_1 = 0$, $\beta_2 = 1$.

Fig. 2 shows the $\beta_1\beta_2$ field with two sets of hyperbolae (13) corresponding to $n = 10$ and $n = 100$ and to $H_2 = \cdot 2$, $\cdot 4$, $\cdot 6$, $\cdot 8$ and $1 \cdot 0$, and with the set of straight lines (16) corresponding to the same values of H_2. We see that the hyperbolae for $n = 100$ are close to the corresponding straight lines except for very small values of H_2.

Considering this figure we see that although the very high values of H_2 fall on the whole in the area of J- and U-curves, this constant may yet have quite considerable values for populations of types commonly met with, and it is clear that one must be very cautious in drawing conclusions from Student's test, if there be no good reason for supposing that the original frequency distribution is normal. We see further that this is not a case where the position is improved in large samples.

The difficulties involved in the application of Student's test may be illustrated also in the following way. Assume that we are dealing with a sample from a population with $\beta_1 = \cdot 27$ and $\beta_2 = 3 \cdot 5$ *. Suppose further that the variance of the sample shows a deviation from its mean equal to $2\sigma_{\sigma^2}$; for the deviation of the mean in the sample, we can take two cases, first—that it is equal to $+ 2\sigma_{\bar{x}}$ and second—that it is equal to $- 2\sigma_{\bar{x}}$. If we express the deviations of both the variates in common units, for instance in terms of $\sigma_x = \sqrt{\frac{\mu_2}{n}}$, we shall have

$$\sigma = \sqrt{n - 1 + 2\sqrt{\frac{n-1}{n}}\,[(n-1)\beta_2 - n + 3]}, \qquad \bar{x} = \pm 2,$$

where σ is the standard deviation of the sample, and $\bar{x}$ the deviation of the mean of the sample from the mean of the population. If now we do not know the moments of the sampled population but only σ and the position of the sample mean, we might use Student's method to test the probability of drawing such a sample from a population with mean at distance + 2 (or − 2) from the sample mean. This we should do by calculating in *both* cases the same ratio, viz.

$$z = \frac{|\bar{x}|}{s} \text{ in Student's notation, where } \sigma = s, \text{ or in that of Fisher†,}$$

$$t = \frac{\sqrt{n-1}\,|\bar{x}|}{s} = \frac{|\bar{x}|}{\left\{1 + 2\sqrt{\frac{1}{n}\left(\beta_2 - \frac{n-3}{n-1}\right)}\right\}^{\frac{1}{2}}} \quad \text{.....(17).}$$

If for instance $n = 11$, we find that $z = \cdot 45$ and $t = 1 \cdot 42$.

* These figures correspond to a set of 200 yields of a serial in a field experiment described by Gorski and Stefaniow in *Rocznik Nauk Rolniczych*, 1917.

† For tables of z, see *Biometrika*, Vol. VI. p. 19 (revised and corrected as Table XXV in Pearson's *Tables for Statisticians and Biometricians*, 2nd Edition), and for t, Fisher's *Statistical Methods for Research Workers*, 1925.

Now let us consider the two cases from the point of view of the correlation between the variance and the mean. We shall see below that the difference between $H_k^2 - H_2^2$, where $k > 2$, is of the order $\frac{1}{n^2}$, therefore we can assume that the actual regression line is well fitted by the second order parabola. If we calculate its ordinates corresponding to the two points $\bar{x} = \pm 2$, we shall have

$$u(-2) = -\cdot 564; \qquad u(+2) = \cdot 688.$$

Taking the ratio of the deviations of the actual value of σ^2 in the sample, from these two means of different arrays, to the corresponding mean array-standard deviation, we have

$$\frac{\sigma^2 - \bar{\sigma}^2 - u(-2)}{\sqrt{1 - H_2^2}} = 2\cdot 7; \qquad \frac{\sigma^2 - \bar{\sigma}^2 - u(+2)}{\sqrt{1 - H_2^2}} = 1\cdot 4,$$

and we see at once that the probability of obtaining the two samples ($\bar{x} = -2$, $\sigma^2 = \bar{\sigma}^2 + 2$) and ($\bar{x} = +2$, $\sigma^2 = \bar{\sigma}^2 + 2$) from the same population is very different, although Student's test gave us the same figure, $t = 1\cdot 42$, for both.

The method described can be considered as a general method of testing whether a sample is likely to have been taken from a population with *a given distribution*, at least, *with given* β_1, β_2 *and* μ_2.

As we have seen, it consists in comparing the deviation $\frac{\sigma^2 - \bar{\sigma}^2 - u(\bar{x})}{\sigma_{\sigma^2}}$ with its standard deviation $\sqrt{1 - H_2^2}$, that is to say, in calculating the ratio

$$L = \frac{\Delta\sigma^2 - u(\bar{x})}{\sqrt{1 - H_2^2}}$$

$$= \frac{(2n + \beta_2 - \beta_1 - 3)\left[\sqrt{(n-1)\beta_2 - n + 3}\,\Delta\sigma^2 - \sqrt{(n-1)\beta_1}\,\bar{x}\right] - \sqrt{n(n-1)}(\beta_2 - \beta_1 - 3)\left(\bar{x}^2 - \sqrt{\frac{\beta_1}{n}}\,\bar{x} - 1\right)}{n\sqrt{2\,(2n + \beta_2 - \beta_1 - 3)\,(\beta_2 - \beta_1 - 1)}}$$

.....(18),

where $\bar{x}$ is the deviation of the mean in the sample from the mean of the population measured in terms of $\sqrt{\frac{\mu_2}{n}}$, μ_2 being the variance of the population; $\Delta\sigma^2$ is the deviation of the variance in the sample from its mean value $\frac{n-1}{n}\mu_2$ measured in terms of

$$\sigma_{\sigma^2} = \frac{\mu_2}{n}\sqrt{\frac{n-1}{n}\left((n-1)\beta_2 - n + 3\right)};$$

n is the size of the sample; β_1 and β_2 refer to the sampled population. If L is nearly 3, we may consider it as improbable that the sample has been taken from the population considered; if however $L < 2$ such an assumption will not be in disagreement with the observed mean and standard deviation of the sample. Unfortunately such cases in which we know the constants μ_2, β_1 and β_2 of the

sampled population are very seldom met with. When as usual they are unknown the results of this paper serve to emphasize the caution which it is necessary to exercise in drawing conclusions from the contents of small samples*.

3. We shall now prove that the difference $H_k^2 - H_2^2$, where $k > 2$, is at least of the order $\frac{1}{n^3}$. We shall do it by showing that any term $V_k^2/\Delta_{k-1}\Delta_k$ of the series (35)† is of the order $\frac{1}{n^3}$. As the calculations necessary for this purpose are easier to do than to describe, we shall avoid the details and simply sketch the general method of procedure.

We have

$$V_k = \begin{vmatrix} q_0, & q_1, & \dots & q_k \\ M_0, & M_1, & & M_k \\ M_1, & M_2, & \dots & M_{k+1} \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ M_{k-1}, & M_k, & & M_{2k-1} \end{vmatrix},$$

$$\Delta_k = \begin{vmatrix} M_0, & M_1, & \dots & M_k \\ M_1, & M_2, & \dots & M_{k+1} \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ M_k, & M_{k+1}, & \dots & M_{2k} \end{vmatrix},$$

where the M's are the moments of the mean of the sample about its mean, measured in terms of $\sigma_{\bar{x}}$.

Let us calculate some first terms of M's and q's which will be clearly polynomials in $\frac{1}{n}$. For this purpose let X be the deviation of the mean in the sample from its mean measured in ordinary units and $[A]$ be the mean value of any

* To avoid possible misunderstanding we may here add the following remarks:

Let us consider the variate $v_k = \sigma^2 - \bar{\sigma}^2 - u_k(\bar{x})$,

where $u_k(\bar{x})$ is the ordinate of the kth order parabola of regression. The variate v_k is a character of the sample in the sense that to every sample there corresponds a perfectly definite value of v_k. Therefore we may calculate the mean $\bar{v}_k$ of v_k and its standard deviation σ_{v_k}. We have, rigorously,

$$\bar{v}_k = 0\,; \quad \sigma_{v_k} = \sqrt{1 - H_k^2},$$

remembering that σ^2 is measured in terms of its own standard deviation.

The proposed method of testing is therefore constructed on the usual principle of comparing the deviation of a variate with the corresponding standard deviation. We notice that if k increases, σ_{v_k} decreases and thus our test becomes more and more accurate.

Of course it would be useful if we knew the value of the standard deviation of v_k for each array corresponding to the different values of $\bar{x}$, because very probably they are not constant. Then it would be possible to consider the variates $v_k(\bar{x})$ corresponding to different values of $\bar{x}$. In any case the proposed L-test seems to be as justified as any other test built on the same principle.

† In my previous paper.

variate A. Then, if $x_1, x_2, \dots x_n$ are the deviations of individuals in the sample from the mean of the population,

$$X^k = \left(\frac{\sum_{r=1}^{n} x_i}{n}\right)^k = \frac{1}{n^k} \sum_{r_1, r_2, \dots r_n} \frac{k!}{r_1!\, r_2! \dots r_n!} x_1^{r_1} x_2^{r_2} \dots x_n^{r_n} \dots\dots\dots(19),$$

where the sum is extended over all combinations of non-negative integral values of r's satisfying the condition

$$r_1 + r_2 + \dots + r_n = k.$$

Accordingly

$$[X^k] = \frac{1}{n^k} \sum_{r_1, r_2, \dots r_n} \frac{k!}{r_1!\, r_2! \dots r_n!} \mu_{r_1} \mu_{r_2} \dots \mu_{r_n} \dots\dots\dots\dots\dots\dots(20).$$

This sum contains several classes of equal terms corresponding each to a single system of values of the r's written in different orders, the number of terms being dependent upon n. We notice that $\mu_1 = 0$ and we assume $\mu_0 = 1$. The most numerous class of non-vanishing terms of (20) corresponds to the following systems of r's:

$$\left.\begin{aligned} r_i &= 2 \quad \left(i = 1, 2, \dots \frac{k}{2}\right) \\ r_j &= 0 \quad \left(j = \frac{k}{2} + 1, \dots n\right) \end{aligned}\right\} \text{ if } k \text{ be even,}$$

and

$$\left.\begin{aligned} r_i &= 2 \quad \left(i = 1, 2, \dots \frac{k-3}{2}\right) \\ r_{\frac{k-1}{2}} &= 3 \\ r_j &= 0 \quad \left(j = \frac{k+1}{2}, \dots n\right) \end{aligned}\right\} \text{ if } k \text{ be odd.}$$

In the first case let $k = 2s$ and in the second $k = 2s + 1$. We have

$$[X^{2s}] = \frac{(2s)!}{2^s\, s!} \frac{n!}{(n-s)!} \frac{\mu_2^s}{n^{2s}} + \dots \left(\text{higher terms in } \frac{1}{n}\right) \dots\dots\dots\dots(21),$$

$$[X^{2s+1}] = \frac{(2s+1)!}{3!\, 2^{s-1} (s-1)!} \frac{n!}{(n-s)!} \frac{\mu_3 \mu_2^{s-1}}{n^{2s+1}} + \dots \left(\text{higher terms in } \frac{1}{n}\right) \dots(22).$$

Proceeding in this way and dividing finally the terms of $[X^{2s}]$ by $\left(\frac{\mu_2}{n}\right)^s$ and the terms of $[X^{2s+1}]$ by $\left(\frac{\mu_2}{n}\right)^{\frac{2s+1}{2}}$, we find

$$M_{2s} = \frac{(2s)!}{2^s\, s!} \left[1 + \frac{s(s-1)}{n} \left(\frac{\beta_2}{6} + \frac{(s-2)\beta_1}{9} - \frac{1}{2}\right) + \frac{1}{n^2}(\dots) + \dots\right] \dots(23),$$

$$M_{2s+1} = \frac{(2s+1)!\sqrt{\beta_1}}{3!\,(s-1)!\, 2^{s-1}\sqrt{n}} \left[1 + \frac{s-1}{n}\left(\frac{\beta_3}{10\beta_1} + \frac{(s-2)\beta_2}{3!} + \frac{(s-2)(s-3)\beta_1}{(3!)^2} - \frac{s}{2}\right) + \frac{1}{n^2}(\dots) + \dots\right]$$
$$\dots\dots\dots(24).$$

In the same way,

$$q_{2s}=\frac{\sqrt{n}}{\sqrt{(n-1)[(n-1)\beta_2-n+3]}}\left[\frac{(2s)!}{2^s(s-1)!}(\beta_2-3)+\frac{(2s)!}{3\times 2^{s-1}(s-2)!}\beta_1+\frac{1}{n}(\dots)+\dots\right] \quad \dots\dots\dots(25),$$

$$q_{2s+1}=\frac{n\sqrt{\beta_1}}{\sqrt{(n-1)[(n-1)\beta_2-n+3]}}\frac{(2s+1)!}{2^s s!}\left\{1+\frac{s}{3n}\left[\frac{\beta_2}{\beta_1}+\frac{3(s-1)}{2}\beta_2+\frac{(s-1)(s-2)\beta_1}{3}-\frac{9s^2+11s+6}{2}\right]+\frac{1}{n^2}(\dots)+\dots\right\} \quad \dots(26).$$

Putting these values into Δ_k, we find that in its expansion the terms independent of n do not vanish and so that if $n\to\infty$ we have

$$\Delta_{2s}\to\prod_{t=1}^{2s}(t!); \qquad \Delta_{2s+1}\to\frac{(2s+2)!}{2^{s+1}(s+1)!}\prod_{t=1}^{s}[(2t)!]^2 \dots\dots\dots\dots\dots(27).$$

As to the expansion of V_k, (where $k\geqslant 3$), not only do the terms independent of n vanish, but the terms with the lowest power of $\frac{1}{n}$, that is with $\frac{1}{\sqrt{n}}$, vanish also. The next power, or $\frac{1}{n}$, in this expansion is the first one. Thus V_k has a common factor $\frac{1}{n}$ and so the kth term of the expansion (35) of my previous paper, for $k\geqslant 3$, has the common factor $\frac{1}{n^2}$. If n be large this can be considered as showing that the second order parabola gives a good fit to the regression line of the variance on the mean of the sample.

In conclusion I wish to express my warm thanks to Dr E. S. Pearson for his help in the English redaction of this paper and for the suggestion to add to it tables and diagrams, the latter of which have been skilfully executed by Miss Ida McLearn of the Galton Laboratory.

TABLE I. *Values of* $H_1=|r|$.

	$n=10$				$n=50$				
$\beta_2\ \beta_1$	·0	·1	·5	1·0	·0	·1	·5	1·0	$\beta_1\ \beta_2$
1·5	·0000	·3721	·8321	—	·0000	·4300	·9615	—	1·5
2·0	·0000	·2860	·6396	·9045	·0000	·3100	·6931	·9802	2·0
2·5	·0000	·2410	·5388	·7620	·0000	·2548	·5697	·8056	2·5
3·0	·0000	·2121	·4744	·6708	·0000	·2214	·4950	·7000	3·0
3·5	·0000	·1917	·4286	·6061	·0000	·1984	·4436	·6274	3·5
4·0	·0000	·1762	·3939	·5571	·0000	·1813	·4055	·5735	4·0
4·5	·0000	·1639	·3665	·5183	·0000	·1680	·3758	·5314	4·5

N.B. Figures above a continuous line correspond to the *U*-shaped, and figures between a continuous and a dotted line to the *J*-shaped curves of Pearson.

TABLE I (*continued*).

β_2 \ β_1	$n=100$ ·0	·1	·5	1·0	$n=1000$ ·0	·1	·5	1·0	β_1 / β_2
1·5	·0000	·4384	·9804	—	·0000	·4463	·9980	—	1·5
2·0	·0000	·3131	·7001	·9900	·0000	·3159	·7064	·9990	2·0
2·5	·0000	·2565	·5735	·8111	·0000	·2580	·5770	·8160	2·5
3·0	·0000	·2225	·4975	·7036	·0000	·2235	·4997	·7068	3·0
3·5	·0000	·1992	·4454	·6299	·0000	·1999	·4470	·6322	3·5
4·0	·0000	·1820	·4069	·5754	·0000	·1825	·4081	·5772	4·0
4·5	·0000	·1685	·3769	·5330	·0000	·1690	·3779	·5344	4·5

TABLE II. *Values of* H_2.

β_2 \ β_1	$n=10$ ·0	·1	·5	1·0	$n=50$ ·0	·1	·5	1·0	β_1 / β_2
1·5	·4103	·5754	1·0000	—	·2055	·4827	1·0000	—	1·5
2·0	·2075	·3663	·7132	1·0000	·0985	·3284	·7088	1·0000	2·0
2·5	·0863	·2624	·5665	·8070	·0404	·2593	·5754	·8148	2·5
3·0	·0000	·2127	·4804	·6882	·0000	·2215	·4962	·7035	3·0
3·5	·0669	·1990	·4286	·6100	·0313	·2000	·4436	·6281	3·5
4·0	·1216	·2075	·3987	·5571	·0579	·1885	·4065	·5735	4·0
4·5	·1677	·2269	·3836	·5215	·0791	·1836	·3795	·5321	4·5

β_2 \ β_1	$n=100$ ·0	·1	·5	1·0	$n=1000$ ·0	·1	·5	1·0	β_1 / β_2
1·5	·1476	·4659	1·0000	—	·0474	·4492	1·0000	—	1·5
2·0	·0702	·3235	·7150	1·0000	·0224	·3169	·7072	1·0000	2·0
2·5	·0287	·2588	·5764	·8156	·0091	·2583	·5773	·8164	2·5
3·0	·0000	·2225	·4981	·7053	·0000	·2235	·4998	·7069	3·0
3·5	·0244	·2000	·4454	·6303	·0071	·2000	·4470	·6322	3·5
4·0	·0406	·1856	·4074	·5754	·0129	·1829	·4082	·5772	4·0
4·5	·0563	·1766	·3787	·5333	·0179	·1698	·3780	·5344	4·5

N.B. Figures above a continuous line correspond to the *U*-shaped, and figures between a continuous and a dotted line to the *J*-shaped curves of Pearson.

TABLE III.

Values of $100 \times \frac{H_2^2 - H_1^2}{H_2^2}$.

β_2 \ β_1	$n=10$ ·0	·1	·5	1·0	$n=50$ ·0	·1	·5	1·0	β_1 \ β_2
1·5	100·0	58·2	30·8	—	100·0	20·6	7·5	—	1·5
2·0	100·0	39·0	19·6	18·2	100·0	10·9	4·4	3·9	2·0
2·5	100·0	15·7	9·5	10·8	100·0	3·5	2·0	2·2	2·5
3·0	?*	·5	2·5	5·0	?*	·1	·5	1·0	3·0
3·5	100·0	7·3	·0	1·3	100·0	1·6	·0	·3	3·5
4·0	100·0	27·9	2·4	·0	100·0	7·4	·5	·0	4·0
4·5	100·0	47·8	8·7	1·2	100·0	16·2	1·9	·2	4·5

β_2 \ β_1	$n=100$ ·0	·1	·5	1·0	$n=1000$ ·0	·1	·5	1·0	β_1 \ β_2
1·5	100·0	11·4	3·9	—	100·0	1·3	·4	—	1·5
2·0	100·0	5·7	2·2	2·0	100·0	·6	·2	·2	2·0
2·5	100·0	1·8	1·0	1·1	100·0	·2	·1	·1	2·5
3·0	?*	·0	·2	·5	?*	·0	·0	·1	3·0
3·5	100·0	·8	·0	·1	100·0	·0	·0	·0	3·5
4·0	100·0	3·9	·2	·0	100·0	·4	·0	·0	4·0
4·5	100·0	8·9	1·0	·1	100·0	1·0	·1	·0	4·5

N.B. Figures above a continuous line correspond to the *U*-shaped, and figures between a continuous and a dotted line to the *J*-shaped curves of Pearson.

* The value of $100\,(H_2^2 - H_1^2)/H_2^2$ at $\beta_1=0$, $\beta_2=3$ depends on the manner in which we approach this point. If we take $m=(\beta_2-3)/\beta_1$, then we have

$$100\,(H_2^2 - H_1^2)/H_2^2 = \beta_1 \Big/ \left(\beta_1 + \frac{\beta_1}{m-1} + \frac{2n}{(m-1)^2}\right) = 100 \text{ if } m \to \infty,$$

i.e. if we approach along a curve tangential at $\beta_2=3$ to $\beta_1=0$; but if m be finite, i.e. if we approach along any curve passing through this point ($\beta_1=0$, $\beta_2=3$) but not tangential to $\beta_1=0$, then

$$100\,(H_2^2 - H_1^2)/H_2^2 = 0.$$

MISCELLANEA.

The Use of Biometric Methods applied to Craniology, being a critique of Professor Gordon Harrower's "A Study of the Hokien and the Tamil Skull." *Trans. R. S. Edinburgh,* Vol. LIV. Part 3, No. 13. (Issued July 6, 1926.)

By G. M. MORANT, D.Sc.

THE craniometric methods which are used to-day by workers in the Biometric Laboratory are, with some additions and slight modifications, those which were devised in the early days of Biometry, and since Miss Fawcett's paper in the first volume of *Biometrika* several thousand skulls have been measured in precisely the same way. Those methods were drawn up by Professor Karl Pearson with the object of providing data which would be comparable with the measurements of both French and German anthropologists—a consideration almost entirely neglected in earlier British craniometric work—and the fact that many different workers have employed them is a sure guarantee of their practicability. A great advance in descriptive technique was made when Crewdson-Benington published the first type contours of a racial series of skulls and Professor Pearson's more recent coefficient of racial likeness is proving an indispensable aid to the statistical student. But the work involved in providing such complete descriptions and comparisons of adequately long series of crania is very great, and the accumulation of that sufficient bulk of evidence, which is required before statistically reliable conclusions can be drawn, proceeds at a rate which is painfully slow from the individual's point of view. It is with a peculiar pleasure, therefore, that we welcome the publication of a paper discussing a collection of skulls by aid of the methods we are ourselves using. This study of Dr Harrower's is, we believe, the first not issued from the Biometric Laboratory which gives type contours and coefficients of racial likeness and which adopts the measurements used there. The skulls dealt with are those of unclaimed coolies collected in Singapore immediately after death. There are 36 ♂ Chinese from the southern province of Foo-kien and 35 ♂ Tamils from the south of India and these samples, though small, should provide good first approximations to the racial types. In many ways which might seem unimportant to anyone not well acquainted with the technique of measurement Professor Harrower has unfortunately deviated from biometric usage. The usual index letters denoting measurements were adopted but with certain alterations and one or two unusual definitions are particularly likely to lead to confusion. The writer says, "I have followed the greater number of workers in making the measurement (the nasal height) from nasion to the most prominent point on the nasal spine," but we do not know any other modern worker who follows that practice. Such a measurement is certainly not comparable with any taken in the Biometric Laboratory, or with Broca's to the base of the spine or with the Frankfurt height, and it is not to be found in Martin's detailed list. The capacities found by shot are not comparable with those given by Macdonell's seed and water method. But a more serious discordance is found in the definitions of orbital diameters. It has been the custom of all craniometricians to take an orbital width along the major axis of the roughly elliptical cavity, though different investigators have employed different terminal points on the inner margin, and the orbital height perpendicular to this width. But Professor Harrower defines the height he has taken to be the greatest in a vertical plane—presumably perpendicular to the Frankfurt horizontal—and then the breadth is taken from the dacryon at right angles to the height. The latter is certainly not the same as Broca's dacryal breadth and, as far as we are aware, both this height and breadth have not been previously employed in measuring a cranial

series. In protesting against what he calls "non-essential measurements" the writer of this paper remarks that, "the actual difference between the basi-apical (our H) and basi-bregmatic (our H') heights is infinitesimal compared to the range of variation of either measurement in a homogeneous series," and hence he only measures the latter and would apparently consider it comparable with the "basi-apical" height given by other workers. Such a statement displays an entire lack of appreciation of the effort that has been made to make these comparisons rigorously legitimate. The kind of fallacy to which it opens the door may be better illustrated in the case of orbital measurements. The question is, of course, whether the mean differences due to the use of different methods of measurement are infinitesimally small or not compared with the probable errors *of the means*. There are two orbital widths commonly taken in this laboratory, Broca's from the dacryon (our O_1') and the other from the pencil line completing the inner orbital margin (our O_1)*. The second always exceeds the first and the ♂ mean differences for series on which both have been taken are: Tibetan A 2·4 (37), Burmese A 4·6 (41), 1st Dynasty Egyptians 2·8 (33), Farringdon Street English 2·5 (81) and for the last series the standard deviation of O_1 is 1·56 ± ·08 and of O_1' 1·67 ± ·09. It would obviously be absurd to suppose that the mean orbital measurements found by the two different methods are directly comparable. Considerations such as these have led to the conclusion that the measurements of different workers may never be compared unless they were taken in precisely the same way: the differences due to the use of different methods of measurement are by no means infinitesimally small compared with the probable errors of the means, the inter-racial differences and the inter- and intra-racial standard deviations. Hence we are obliged to duplicate measurements in order to provide comparable data. We can only suppose that conclusions drawn from the direct comparison of measurements that are not identical, as in this paper, are, in all probability, entirely fallacious. In Table II the means of the series of Chinese and Tamil skulls are compared with those of 14 other races and there is no indication of the papers in which the quoted measurements were first published. That omission is all the more to be regretted as the numbers of skulls on which the means were based are not given and the student who wishes to use them in computing coefficients of racial likeness, or for other statistical purposes, must refer to the original studies. The means given for Turner's Scottish skulls certainly represent a heterogeneous population. The Hokien facial index, wrongly given as 76·5, should be 73·8; the Tamil transverse arc, wrongly given as 321·6, should be 313·2 and the Tamil occipital index, wrongly given as 63·95, should be 62·2. In computing coefficients of racial likeness Professor Harrower used the standard deviations of the long series of 26th—30th Dynasty Egyptian skulls measured and preserved in the Biometric Laboratory but no acknowledgement is made of that fact, and it would not be clear to anyone not well acquainted with the craniometric work done there that the greater number of the coefficients given are quoted from the pages of *Biometrika*. The sagittal and transverse type contours—the horizontal types not being provided—were prepared in the customary way with slight modifications. On the former the point of maximum subtense from the nasion-inion line is given instead of the point of maximum subtense from the glabella-inion line and on the latter the auricular points were located further inside the passages than is usual.

The only comparative Asiatic material used in this paper is taken from the papers of Tildesley†—misspelt Tyldesley throughout—and Morant‡. Professor Harrower was apparently not acquainted with a later biometric study of the Nepalese skull in which reduced mean skull measurements for a number of other Oriental races are given§. He says (p. 573): "China covers a vast expanse, much larger than most people realise, and its inhabitants include a large variety of branches which vary widely in craniological features....The dissimilarity between the Cantonese Chinese and the Hylam Chinese skull is, I believe, as great as that between the

* See *Biometrika*, Vol. I. p. 430 and Vol. VIII. pp. 311—312.

† M. L. Tildesley, "A First Study of the Burmese Skull," *Ibid.* Vol. XIII. 1921, pp. 176—262.

‡ G. M. Morant, "A First Study of the Tibetan Skull," *Ibid.* Vol. XIV. 1923, pp. 193—260.

§ G. M. Morant, "A Study of certain Oriental Series of Crania including the Nepalese and Tibetan Series in the British Museum (Natural History)," *Ibid.* Vol. XVI. 1924, pp. 1—104.

Scottish and the German skull." But no comparisons with earlier studied series of Chinese skulls and, indeed, no evidence of any kind are brought forward in support of these statements. In the last-mentioned paper the mean measurements of three fairly long series of Chinese skulls from different parts of the country are given. Though not identical, they proved so similar that the conclusion that all could be considered local varieties of a single type seemed to be quite justified. It was surprising to find that samples drawn from such an enormous and widely scattered population should indicate racial homogeneity and the comparison with the Hokien series is of particular interest as it may confirm or refute Professor Harrower's statement. The coefficients of racial likeness are given in the table below and it may be remarked that the lowest found by Professor Harrower between the Hokien Chinese and an Asiatic race was 5·57 ± ·17 (corrected probable error) with the Tibetans of the A type though that is rather higher than it should be owing to the fallacious comparison of orbital measurements.

Coefficients of Racial Likeness between Series of Chinese Male Skulls.

		Northern Chinese Koganei (69·4)*	Hokien Chinese (36·0)	Northern Chinese (36·4)	Southern Chinese (52·5)
Northern Chinese Koganei (69·4)	All Characters	—	1·54 ± ·21	2·56 ± ·19	6·15 ± ·19
	Indices and Angles	—	0·35 ± ·61	1·58 ± ·32	5·45 ± ·32
Hokien Chinese (36·0)	All Characters	1·54 ± ·21	—	1·62 ± ·17	1·76 ± ·18
	Indices and Angles	0·35 ± ·61	—	2·10 ± ·26	1·24 ± ·29
Northern Chinese (36·4)	All Characters	2·56 ± ·19	1·62 ± ·17	—	0·59 ± ·17
	Indices and Angles	1·58 ± ·32	2·10 ± ·26	—	1·00 ± ·26
Southern Chinese (52·5)	All Characters	6·15 ± ·19	1·76 ± ·18	0·59 ± ·17	—
	Indices and Angles	5·45 ± ·32	1·24 ± ·29	1·00 ± ·26	—

* The numbers in brackets are the mean numbers of skulls available for the characters used in computing the coefficients.

All the coefficients except those between Koganei's Northern Chinese and the Southern Chinese are of the order of those found between local varieties of the same race or between very closely allied races, so comparison with the Hokien material enables us to re-affirm the belief that the immense population of China belongs predominantly to a single racial type. Making the kind of comparison Professor Harrower suggests between a British (Farringdon Street English (95·6)) and a German (Ranke's *Altbayerisch* (77·0)) ♂ series we find coefficients of 43·92 ± ·19 for 23 characters and 58·97 ± ·33 for 7 indices and angles, results of a widely different order. The longest series of Tamil skulls previously measured appears to have been a short one of 13 ♂ specimens collected by the brothers Sarasin*. The coefficients between the Tamils from Ceylon (12·4) and Harrower's series collected in Singapore (35·0) are 2·25 ± ·19 for 21 characters and 3·54 ± ·32 for 8 indices and angles, but it would be unwise to conclude that there is a real difference between the island and mainland peoples since the samples are so small. The latter series gives coefficients of 5·66 ± ·18 with Veddah means† (24·4) of 27 characters and 4·22 ± ·30

* Paul and Fritz Sarasin, *Ergebnisse naturwissenschaftliche Forschungen auf Ceylon.* Wiesbaden, 1892–4.

† Given in *Biometrika*, Vol. XVI. p. 48 and reduced from the measurements of P. and F. Sarasin (*op. cit.*) with the addition of a few others.

when 9 indices and angles are compared. With the reduced Maravar (Dravidian) means* (25·9) the Tamils (35·0) give coefficients of 5·82 ± ·23 for 16 characters and 2·24 ± ·67 for 3 (!) indices. As these are the lowest coefficients that can be found, little can be said about the affinities of the Tamil skull and no craniological classification of the Dravidian peoples of India can be attempted while the material is so meagre.

Commenting on the use of the coefficient of racial likeness, Professor Harrower says: "a skull may be an exact replica of another, but, say, 10 per cent. smaller in every measurement. The C.R.L. between the two skulls would therefore be much greater than the C.R.L. between two skulls which had half the measurements exactly equal, one-fourth 10 per cent. above and the remaining fourth 10 per cent. below: skulls which would be entirely different in general outline" (p. 586). We may remark, in the first place, that no one has ever suggested finding coefficients between single skulls and that if conditions similar to these hypothetical ones were found for mean types—and from what little we know of inter-racial correlations the probability of finding such would seem to be very small—then it is by no means evident why the first coefficient should be much greater than the second. Indeed, the first might well be smaller than the second. The separate computation of coefficients for indices and angles only was, of course, suggested to provide a measure of differences of shape apart from differences of size. One or two minor points may be noticed. It is contrary to biometric usage to give standard deviations without probable errors as on p. 581. The frontal index defined on the previous page should be preceded by 100 and the legitimacy of its use may be questioned as the median sagittal arc from nasion to bregma has not the form of a catenary if the superciliary ridge is at all well developed. We fail to find any evidence adduced to support the statement—made on p. 597—that "the Tibetan A group is hybrid and the Hokien is a pure type," and it is unworthy of a scientific study to draw conclusions such as the following, involving theories which no one has ever attempted to demonstrate. "Intellectually," we are told, "as judged by his cranial capacity, he (the Hokien) is capable of attaining to the highest standard, but the very narrow frontal region (giving the lowest value of all the groups compared) indicates that his powers of initiative and intelligence are relatively low" (p. 597). We are sorry to see such undisciplined theorising cheek by jowl with valuable biometric measurement. Anthropologists are indebted to Professor Harrower, however, for having provided a craniological record which will be of some permanent value.

* *Biometrika*, Vol. xvi. p. 28.

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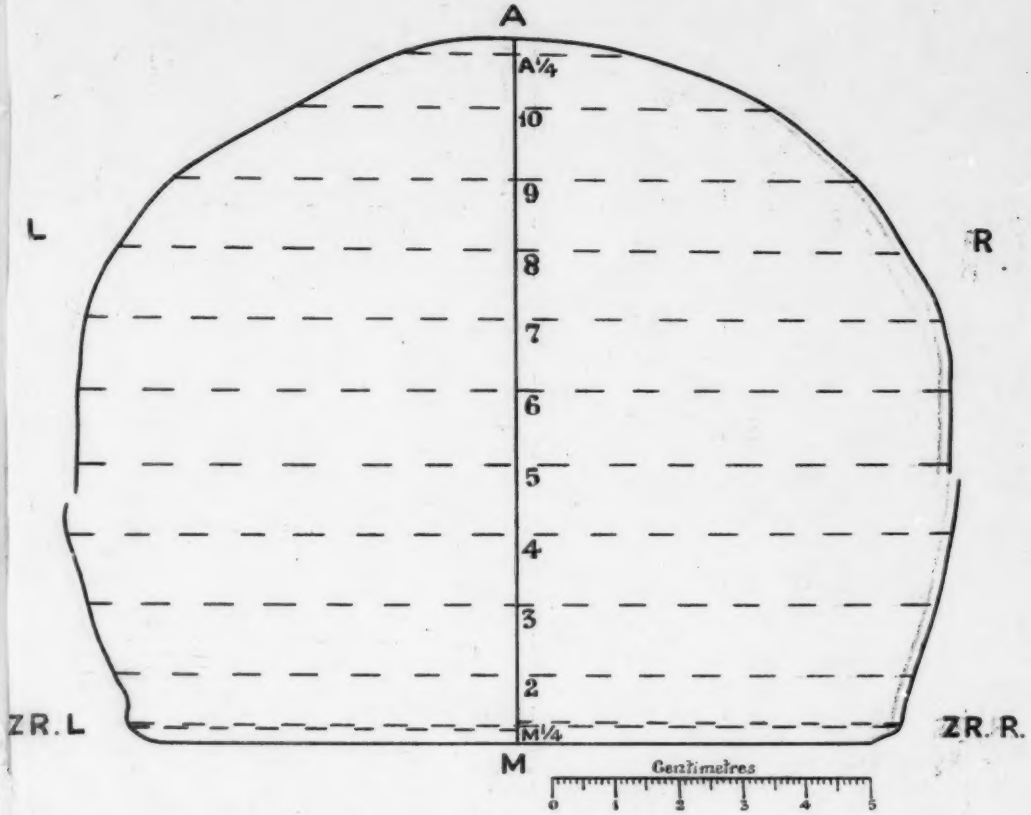
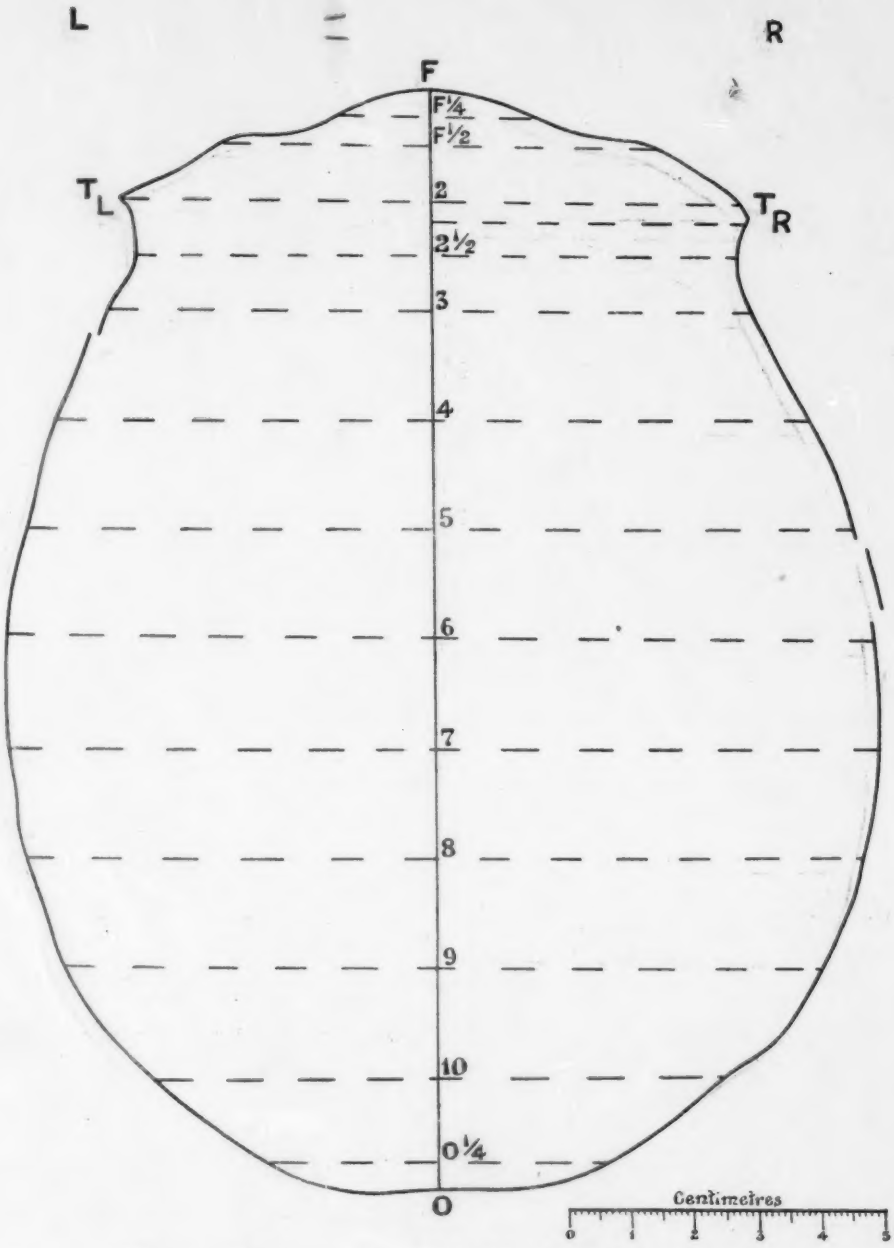


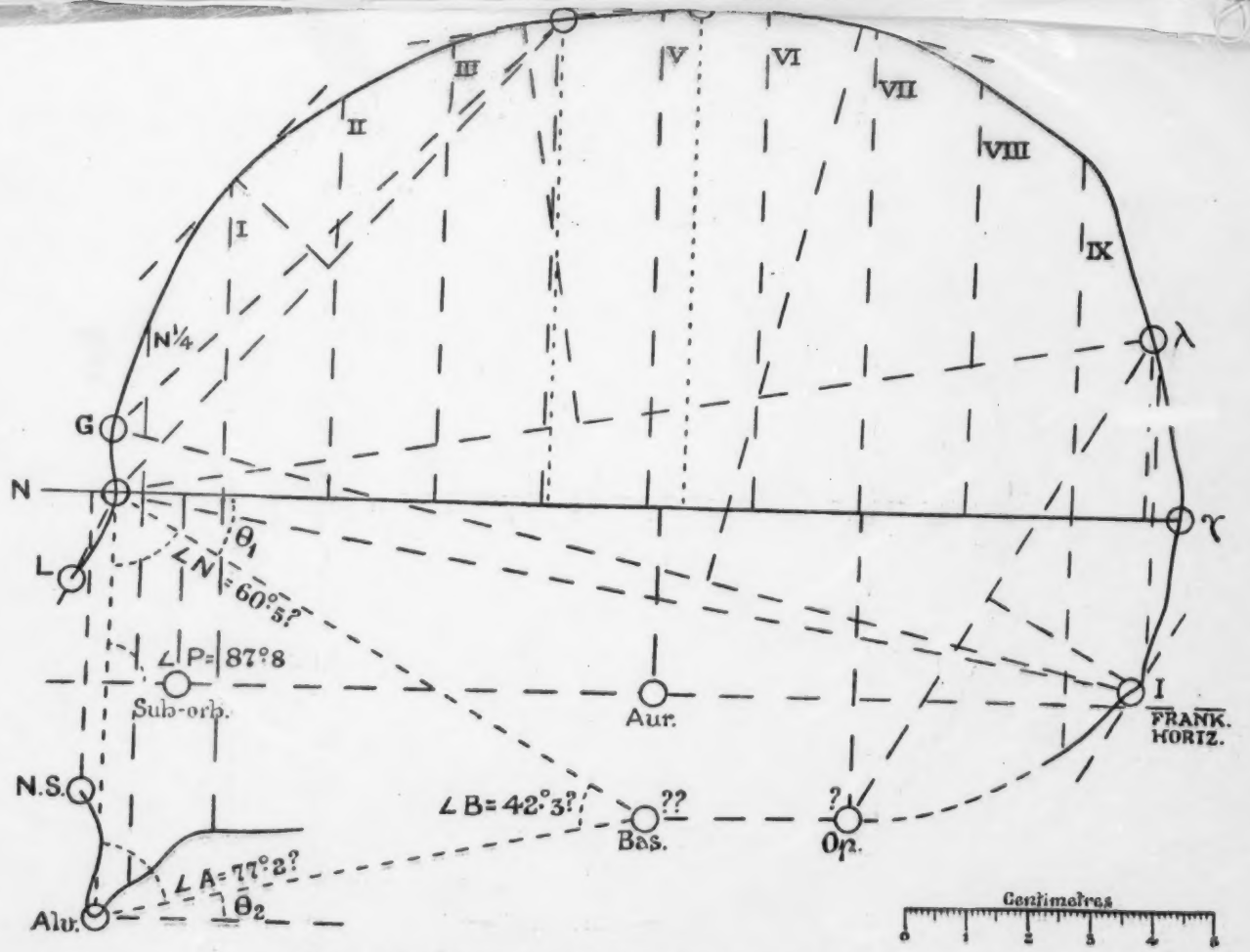
George Buchanan's Skull, Transverse Contour.

Fig. II.



George Buchanan's Skull, Horizontal Contour.

Fig. I.



George Buchanan's Skull, Sagittal Contour.

Fig. III.

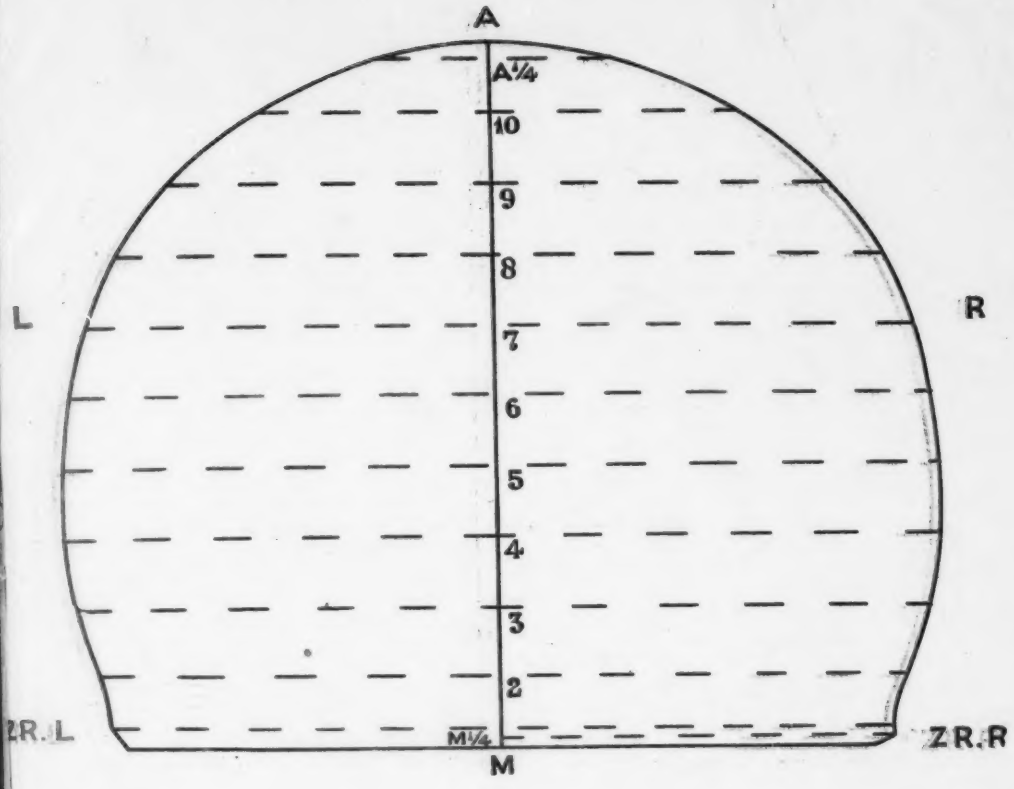


FIG. I 17th Century Londoners ♂ Transverse Type Contour.

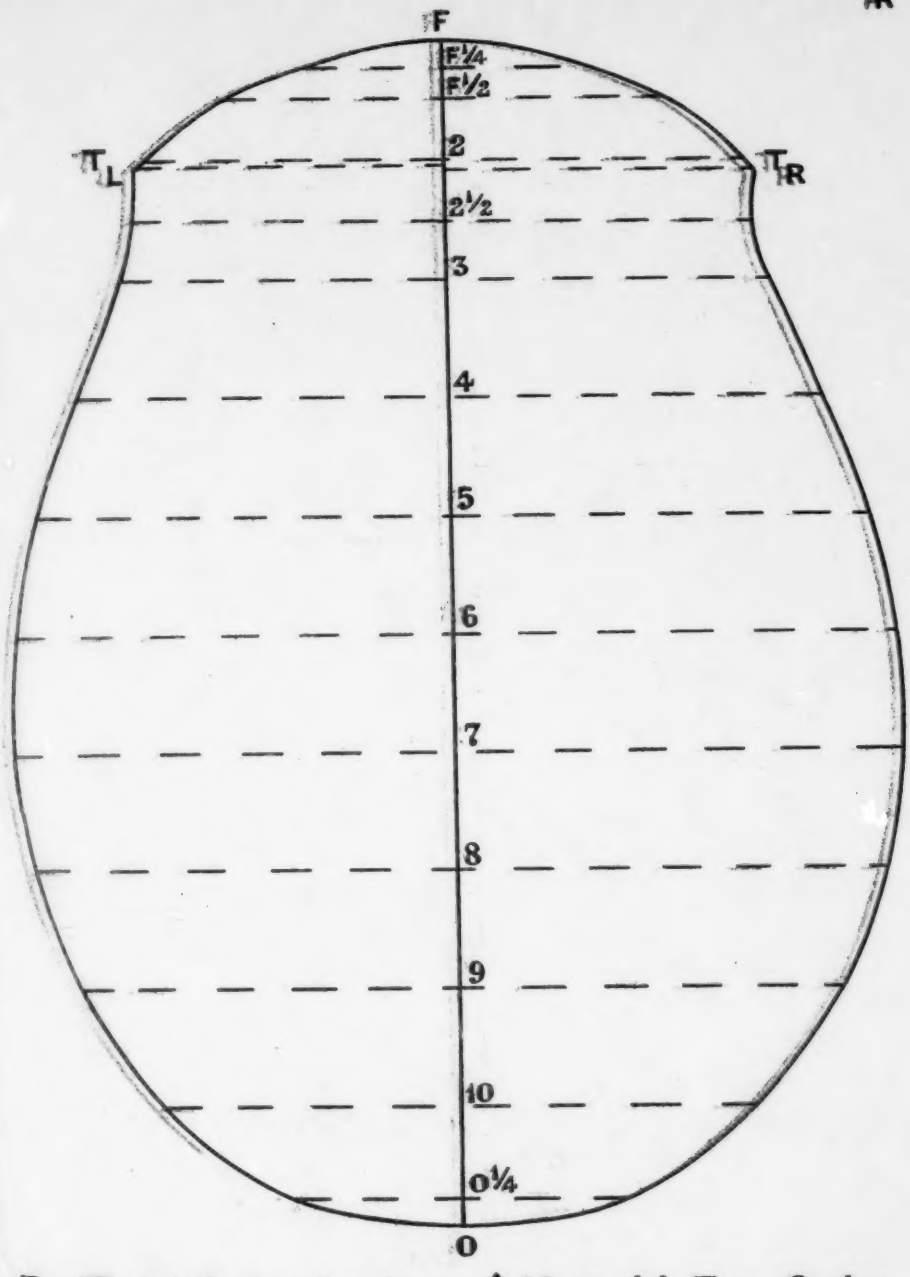


FIG. III 17th Century Londoners ♂ Horizontal Type Contour

β
γ

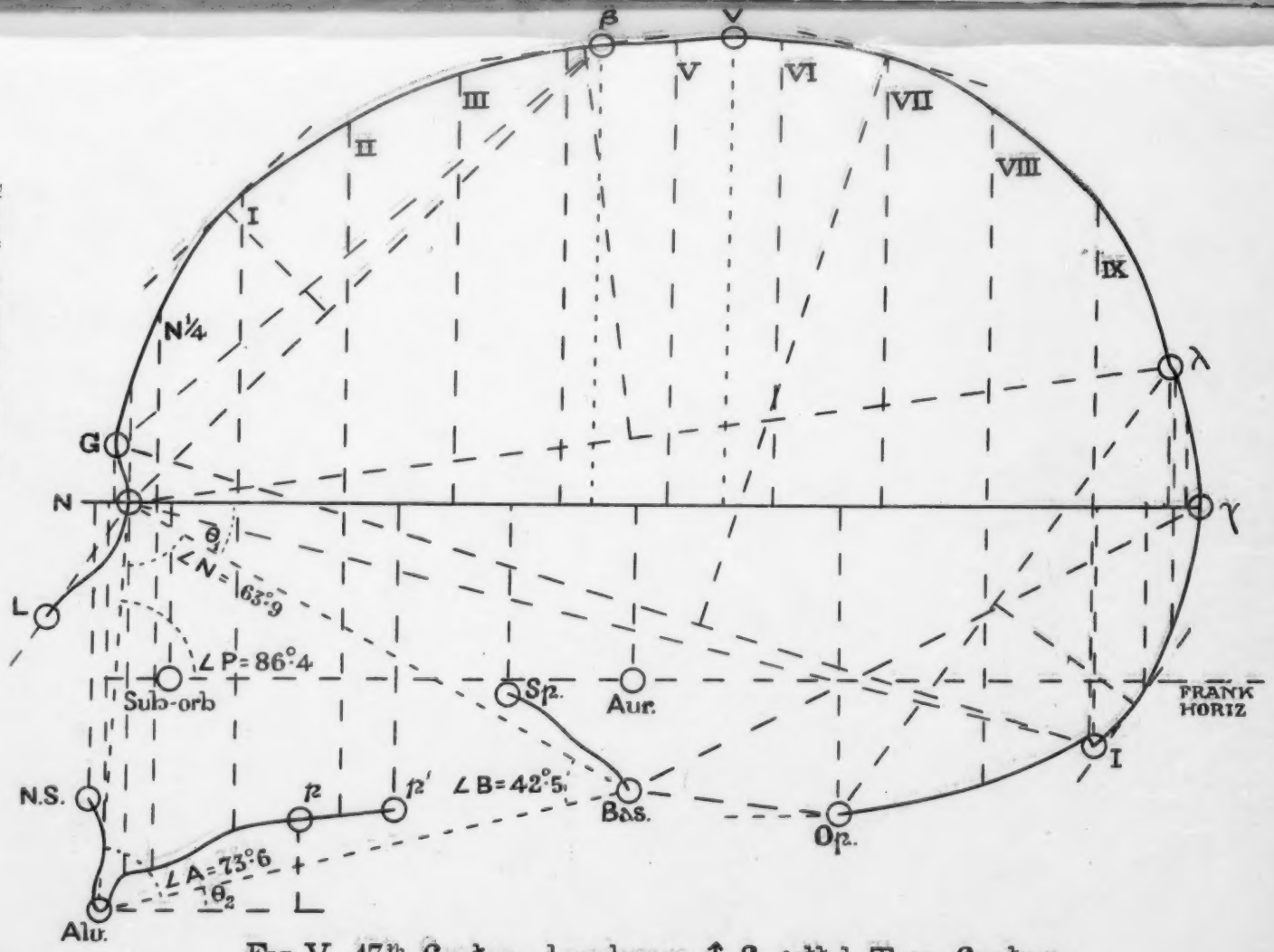


FIG. V 17th Century Londoners ♂ Sagittal Type Contour.

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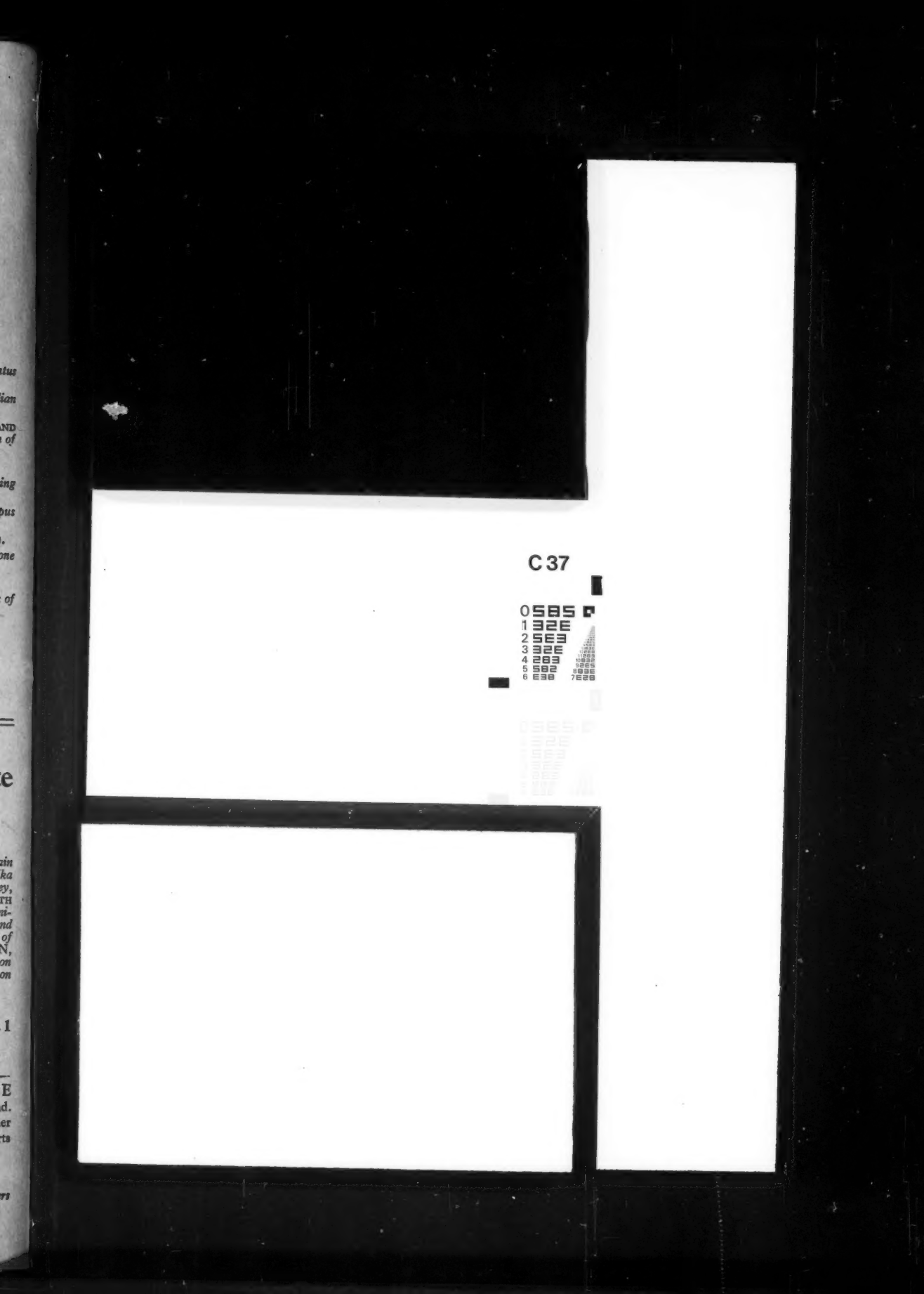
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